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«MARTINGAL ORDERING» OF SHARKOVSKY FOR A SIMPLE LOGISTICAL MAPPING

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*The present work contains the results of the analysis of behavior of a single-parameter family $f_\lambda(x)$ defined by the continuous difference equation that has an attracting periodic orbit. It is demonstrated that all probabilistic f_λ-invariant measures are singular with respect to Lebesgue measure dx, and the iterations $f_\lambda^n{}_* dx$ converge in weak *-topology to a discrete invariant measure. It is supposed that this situation is typical from the topological point of view. This statement is proved for the asymptotic behavior where only transitions to the nearest neighbors are allowed (in the sense of behavior of random trajectories). The classical example for the physical problems is the Wiener measure, but the random processes are Gaussian as a rule in this case.*

Keywords: difference equation, counting measure, martingal, filtration, attractor

Introduction

It is well known that (see, for example, [1,2]) if one parameter family $f_\lambda(x)$ is defined by the continuous difference equation for random trajectories X_t

$$X_{t+1} = \lambda X_t \left(1 - X_t\right),\ t > 0,\ 0 < \lambda < 4, \qquad (1)$$

and transforms some closed interval (0,1) to self for each $x_t \in [0,1)$, so there exists a family of parameters $\lambda \in \Lambda \subset R^1$ such that the set of trajectories (or orbits) of one-dimensional systems defined by the map $f_\lambda(x) = \lambda x(1-x)$, $x \in R^1$ is random in some phase space of trajectories $x_t \in X$ (for example, in the metric of Skorohod [3]). The statement follows from a well-known result of M.V. Jakobson. From the metric point of view, the set $\{\lambda : f_\lambda\}$ has an invariant finite measure μ_λ absolutely continuous with respect to a Lebesgue measure dx ($\mu_\lambda <$ dx) and a positive measure in Λ. So there exists a probabilistic f_λ-invariant measure. By the theorem «about reconstruction» by Kolmogorov [4], measure μ_λ is «one-to-one» to some of simple exclusion processes with translation invariant transition (Markovian process) which preserves stochastic order and has a family of equilibrium measures $\{\gamma^\lambda\}$ indexed by a continuous parameter ranging from 0 to 4 (see also [5]).

Set Λ has a positive measure for the following classes of maps [2]:

1) $f_\lambda(x) = \lambda f(x)$, where $0 < \lambda < 4$ and $f(x)$ is a function C^3 near the quadratic map $x(1 - x)$;

2) $f_\lambda(x) = \lambda f(x)$ (mod1), where f is C^3, $f(0) = f(1) = 0$ and f has a unique nondegenerate critical point in [0,1].

Equation (1) induces an infinite-dimensional discrete dynamical (more exactly, semi-dynamical) system

$$\left\{C^1\left([0,1],I\right)Z^+,S\right\},\ S\varphi = f\circ\varphi \text{ for } \varphi\in C^1\left([0,1],I\right). \quad (2)$$

Here we write x_φ for the solution of (1) with the initial function φ (i.e., $x(t)[0,1) = \varphi(t)$) [6]. The solution $x_\varphi(t)$ results from «joining» the functions $(S^n\varphi)(t)$ and $(S^{n+1}\varphi)(t)$ «tail to tail» at every $n\in Z^+$. In this way, the problem of description on the long-time behavior of solution of equation (1) is reduced to the construction of an attractor for system (2).

This dynamics is represented by a Marcovian evolution with infinitesimal generator L and semigroup (S_t) [6]. We should refer to [5] for a complete description. It is easy to check that $(X_t)_{t\,\geq\,0}$ is a Marcov process defined by f_λ. Consequently, (1) has a unique solution in law concentrated on the right continuous trajectories having left limits [7], such that the probabilistic distributions $P_1\left(x_t^\lambda\in A\right)=P_\lambda\left(x_{t+s}^\lambda\in A\right)$ for all $s > 0$ on each set of cylindrical functions by definition. For $t \to \infty$, the limiting distributions $P_{\mu_\lambda}(x^\lambda(t))$ are asymptotically periodic functions of period p [6–8]. It is true for the semigroup S_t, being a consequence of the well-known theorem of Hille–Iosida [9].

So, the set

$$\lim_{t\to\infty}\mu_0 S^f\left(t\right)=p_1^f,\ldots,p_r^f \quad (3)$$

is constructed of r-points in the class of all invariant measures $\mu\in P$. The limit in (3) can be interpreted in another way. It is well-known that if f_λ has an attracting periodic orbit $\overline{\alpha}=(\alpha_1,\ldots,\alpha_r)$, all probabilistic f_λ-invariant measures are singular with respect to Lebesgue measure dx, and the iterations $f_{\lambda\,*}^n\mathrm{d}x$ converge in weak *-topology to a discrete invariant measure supported by $\overline{\alpha}$. It is probable (but not proved) that this situation is typical from the topological point of view [2]. In the present work, we prove this statement for the asymptotic behavior where only transitions to the nearest neighbors are allowed (in the sense of behavior of random trajectories (1)). We consider a process starting in equilibrium n-space with different parameters α_1, ..., and α_n. We show that for a random initial value $X_{\varphi[0,1)}$ with the initial measure $\gamma^{\alpha_1,\ldots,\alpha_n}$, random solutions of equation (1) for $t \to \infty$ are smooth periodic solutions with measure $\gamma^{\alpha_1,\ldots,\alpha_n}$. Then we prove that this statement is structurally stable for map f^ε, where f_ε^λ is homeomorphic to $f_\lambda^0=f_\lambda$

for some $\varepsilon \le \varepsilon_0$. Further it will be proved that for space of all initial functions $\varphi_\mu \in P$ (where P is a set of all probabilistic measures on X with *-topology: $\mu_n \to \mu$ if and only if $\int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu$ for each $f \in C(X)$; if $\mu \in P$ and P^{x_φ}, $x_\varphi \in X$ that is a Markovian process, then Markovian process x_φ with initial distribution μ, i.e. process x_φ^t with the distribution

$$P^\mu = \int X\, P^\mu \left(\mathrm{d}\eta\right), \tag{4}$$

i.e.

$$E^\mu f\left(X_\varphi^t\right) = \int S^f(t) f \,\mathrm{d}\mu \tag{5}$$

for $f \in C(X)$, the initial distributions $P_{X_\varphi}^\mu(\lambda)$ are tending to some smooth periodical distributions $P_\infty^\mu(\lambda)$ as $t \to \infty$. The ordering of Sivak–Sharkovsky is true for P_∞^μ on the set $\lambda \in \Lambda_1$ [10,11].

The classical example for the physical problems is the Wiener measure, but the random processes are Gaussian as a rule in this case [8]. So the limit in (1) will be defined in *-topology as the limit in L^1 with respect to measure μ [11].

Note that the measure μ_λ is absolutely continuous with respect to measure dx of Lebesgue (we refer to [2]). This fact confirms an analogical statement for Markovian process by Sharkovsky [6]: we prove an analogical result for some more classes of random processes like martingal one [5]. The theorem of Sharkovsky is based on the possibility of construction of «simple» σ-algebra, which is contained by the minimal triangle of closed interval [0,1] constructed by «pseudofiltration» in deterministic sense [9].

In some context, equation (1) can be associated with two problems. First, for $\lambda \in \Lambda \subset \Lambda_1$, there exist «self-to-self» random trajectories for equation (1), and we must to determine the long-time behavior of

$$x_\varphi(t) = f^{[t]}\left(\varphi\{t\}\right), \quad t \in R^+ . \tag{6}$$

Here [•] is for the integer part of a number, {•} is the fractal part with respect to measure μ. And so on we must study the behavior of trajectories in order to add the «white noise». It follows from the last results of [8] that formula «noise + noise» can stabilize chaos in this case. Note that for ε = 0 and λ = 4, the map determined by equation (1) has f_λ-invariant measure with σ-masses. This is the case remarked by Jakobson [2]: if f_λ has an attracting periodic orbit $\overline{\alpha} = (\alpha_1, ..., \alpha_r)$, then all probabilistic f_λ-invariant measures are singular with respect to Lebesgue measure dx, and the iterations $f_{\lambda^*}^n \mathrm{d}x$ converge in the weak *-topology to the discrete invariant measure supported by $\overline{\alpha}$.

It will be proved below that for $\lambda^* \in \Lambda_1$ (in the sense by Jakobson from the metric point of view), there exists an invariant measure of Leab $\mu(\mathrm{d}Z_t)$, where Z_t is so-called λ^2-martingal process in the nonstandard sense [10]. For this measure, iterations $f_{\lambda^*}^{n}\mathrm{d}Z_t$ (with respect to martingal!) converge in SL^2-topology (in nonstandard sense) to an invariant measure $\mu(SL^2)$ such that the ordering of Sharkovsky (for the deterministic maps!) is true.

For $\lambda = \lambda^*$, measure $\mu(SL^2)$ is probabilistic «by Jakobson» [10]. So, in the physical sense, «δ-masses» do not belong to $(\alpha_1,\cdot,\alpha_n)$ trajectories, but they are singular for almost all points $Z_t \in [0,1]$. Thus, $f_\lambda^*\mathrm{d}Z_t$ has the unique invariant measure γ on $\{-1,+1\}^{Z_d}$, Z_d is a triangulation of interval $[0,1]_d$ such that γ is Bernoully product with parameter 1/2. The results are as follows: 1) for $\lambda \in \Lambda/\Lambda_1$, we have Sharkovsky theorem of for a random initial value $X_{\varphi[0,1)}$ in «deterministic sense»; 2) for $\lambda \in \Lambda_1$, at the random initial value $X_{\varphi[0,1)}$, we have stochastic behavior of trajectories on set Λ_1 that is the positive measure in Λ (by Jacobson).

So, for $\lambda \in \Lambda_1$ and deterministic initial value, we have random trajectories which converge by iterations (see (1)) with respect to Leab's measure in *-topology to Sharkovsky ordering for λ^2-martingal process in nonstandard sense.

For λ^2-martingals from the metric point of view (in Skorohod metric) in nonstandard sense, and for Markovian process (by Sharkovsky [1] and Jakobson [2]), we have a «regular embedding» for λ^2-trajectories defined in some weak topology in nonstandard sense (from the topological point of view) to Markov process defined in the strong Skorohod's topology (from the metric point of view).

So, the statement of Jakobson [2] «that is not so from metric point of view» is not true in the defined sense, but it is true by Jakobson.

Let us consider an example, for $\lambda = 3, 2$. In this case, the map f_λ has two repelling fixed points: $z = 0$ and $z = \alpha_1$, and a unique attracting cycle formed by points $\alpha_2^{1,2} = (\lambda + 1 + \sqrt{\lambda^2 - 2\lambda - 3})/2\lambda$. For some random distributions $P(X_{\varphi[0,1)})$, the limiting distributions of solutions of equations (1) have invariant masses of form $\delta(\alpha - \alpha_2^{1,2})$ at points α_2^1 and α_2^2, i.e. at the fixed point of map f_λ. This is an analogy of known result [11] for Browen evolution on [0,1] with attractive screen at 0 and 1 with the difference that the «screens for dynamical system» with the map f_λ are stated at points α_2^1 and α_2^2.

Martingal's measures and their ordering

As known, Wiener constructed the measure on space of Browen's trajectories, Sharkovsky constructed the family of measures in the class of Markovian processes [1]. Such process belongs to the space of random functions that is not compact with respect to distributions. The following ordering is used below: if, intuitively, measure μ_λ defines the frequency of random trajectories belonging to some interval that we have not defined by Lebegue measure dx, that follows from absolutely continuous μ_λ with respect to dx. In the same way, we define intuitively «martingal's interval» $I_t = \mathrm{d}\mu_t$ in the averaging sense by $\int_0^1 X_t \mathrm{d}\mu_t$, where μ_t is a martingal. For example, if μ_t is the white noise, then we have the measure of Wiener. If μ_t is represented by Markovian evolution, a deterministic system possesses trajectories with ω-limit sets consisting of random functions. This phenomenon is called self-stochasticity [1], and the trajectories are self-stochastic.

So, it is possible to use the technique of nonstandard analysis [11], where it is possible to make some estimations for the variation of random functions (or fields by the terminology of statistical physics [10]). Then it is possible to use the Leab's measure, and/or it is partly the case of Dolean measure [10].

Indeed, the method confirms the well-known result of probabilistic measures ordering, i.e. the theorem of Romanenko–Sharkovsky [3]. First we consider the trajectories of random process $X_t(\omega)$ in the space-averaged $\int X_t \mathrm{d}B_t$ in some metric. Here B_t is a random process (generally, it is not connected with X_t), and the «Markovian» distributions of closed interval [0,1].

Lemma 1. Let be $\lambda \in \Lambda$, where Λ is the set of values of parameter λ for which the map $f_\lambda : x \to \lambda x(1 - x)$, $x \in [0,1]$, $0 < \lambda < 4$, has probabilistic f_λ invariant measure μ_λ absolutely continuous with respect to a Lebesgue measure dx. Then for each initial distributions $\mu \in P$, where P is the class of all invariant measures, there exists the identification of all limits

$$\lim_{t\to\infty} \int_t^{t+1} X_t^f \mathrm{d}\mu_t = \lim_{t\to\infty} \int X_t^f \mathrm{d}\mu_t = J\left(f\right) \tag{7}$$

where $J = \{\gamma\}$ is an one-pointing set for $\lambda \in \Lambda$, and a 2^N-pointing set for $\lambda \in \Lambda_n$; n is the minimal common divider for all cycles of map f.

For $\lambda \in \Lambda_1$, the statement of Lemma is reduced to the classical definition of the ergodicity (in the terms of semigroups)

$$\lim_{t\to\infty} \mu S^f(t) = \gamma = J(f) \tag{8}$$

for all $\mu \in P$, where γ is an one-pointing set.

Proof. It should be noted that instead of the trajectories X_t defined by map of equation (1) with some initial distribution $\mu(X_{\varphi[0,1)})$, the trajectories $Z_t = \int_t^{t+1} X_t \mathrm{d}\mu_t$ may be considered formally with respect to some random process with the distribution μ_t (it must be defined).

In the case of possibility of such «integral transform» of a random process for the investigation of the structure of ω-limit set of trajectories $Z_t \in Z$ with logistical map

$$Z_{t+1} = \lambda Z_t\left(1 - Z_t\right) = f_\lambda(Z_t) \tag{9}$$

in some phase space Z, we can deal with an usual transform of R^1 line to itself (or a closed interval to itself), i.e. in R^1.

So, if $Z_t \in P^+(f_\lambda)$ where $P^+(f_\lambda)$ is the set of attractive fixed points of map f_λ, then there exists an usual ordering of Sharkovsky [3] for all points. There exists the set of initial functions $Z_{[0,1)}$ with distribution $P(Z_{[0,1]})$ that are tending to some piecewise-constant asymptotically almost periodic functions with respect to some Dolean measure for $t \to \infty$. The values of the functions belong to set $P^+(f)$.

Of cause, if Z_t is a random function, then we must define the measure. We can say about the existence of periodic trajectories ordered with respect to the period [3], but not about the values of amplitudes of oscillations for the limiting trajectories (functions).

Note that the following transforms are true on the continuous trajectories having left limit of the process Z_t (by supposition), and for each continuous function T with respect to some Borelian triangulation F_t:

$$E^Z\left[f\left(Z_{t+s}\right)\right]=E^Z\left[E\left[f\left(Z_{t+s}\right)|F_t\right]\right]=E^Z\left[E^{Zt}f\left(Z_s\right)\right]=$$
$$=E^Z\left[S\left(s\right)f\right]\left(Z_t\right)=S(t)S(s)f(Z)=S(t+s)f(z). \qquad (10)$$

Then in the sense of the mathematical expectation, the «deterministic ordering by Sharkovsky» is true for the averaged process (see below) and for the distribution of probability. In other words, it is the classical theorem of Sharkovsky [9] (more exactly, the theorem of Sivak–Sharkovsky [7] for one-parameter families of one-dimensional maps). Now we must prove this statement.

Indeed, such mathematical construction is known [10], and we must only formally applicate the results formulated in [10] to our situation.

So, (9) is accompanied by the relation

$$E\left[\left(\int_0^1 X\mathrm{d}\mu\right)^2\right]=E\left(\sum_0^1 X^2M^2\right)=E\left(\int_0^1 X^2\mathrm{d}\gamma_\mu\right), \qquad (11)$$

which is proved in [10] for the case when M is the local λ^2-martingal and $X \in SL(M)$ is the space of S-continuous on the right functions having left limit. It is immediately concluded that the following relations are true:

$$E\left(\int_0^1 X_{t+1}\mathrm{d}M_{t+1}\right)=\lambda E\left(\int_0^1 X_t\mathrm{d}M_t\right)-\lambda E\left(\int_0^1 X_{t+1}\mathrm{d}M_t\right)^2=$$
$$=\lambda E\left(\int_0^1 X_t\,\mathrm{d}M_t\right)-\lambda E\left(\int_0^1 X_t\,\mathrm{d}M_t\right)^2. \qquad (12)$$

They can be written in the form:

$$Z_{t+1}=\lambda Z_t-\lambda\left[Z_t\right], \qquad (13)$$

where Z_t is the variation of Z.

For a structurally stable map, the set of λ is constructed from a finite number of the points $\left|\lambda_j^+\right|<1$ and $\left|\lambda_k^-\right|>1$, $j+k=m$, where m is the number of fixed points of map f_λ. So, these points are one-through-one ordered, i.e. sequentially divide the set of attracting trajectories to some ω-limit set. So variation $E[Z_t]$ converges to zero (or only variation $[Z_t]$ is bounded!) then we can use the operation of mathematical expectation in the limiting relation (11) to state the linearization of map f_λ (in the sense of mathematical expectation) on random trajectories Z_t. Such procedure is frequently applied to the problems of autoregression (see, for example, [4]).

So on, the Sharkovsky's ordering for the difference equation follows:

$$EZ_{t+1} = \lambda EZ_t . \tag{14}$$

We consider long-time behavior of variation $[Z_t]$. We introduce given continuous initial function $h[0,1)$ which satisfies the condition of continuous «joining» of the function: $Z_1 = \lambda Z_0 - \lambda[Z_0]$, where Z_0 is variation of function Z_t at point $t = 0$.

Then we conclude from a simple relation

$$\begin{aligned} Z_{t+2} &= \lambda Z_{t+1} - \lambda\left[Z_{t+1}\right] = \lambda\left(\lambda Z_t - \lambda\left[Z_t\right]\right) - \lambda\left[Z_{t+1}\right] = \\ &= \lambda^2 Z_t - \lambda^2\left[Z_t\right] - \lambda\left[Z_{t+1}\right] = \lambda^2 Z_t - \lambda\left(\lambda Z_t - Z_{t+1}\right) - \lambda\left[Z_{t+1}\right], \end{aligned} \tag{15}$$

that is

$$Z_{t+2} = \lambda Z_{t+1} - \lambda\left[Z_{t+1}\right], \tag{16}$$

that equation (13) is invariant with respect to shift. The process X_t is consequently «one-to-one» for some probabilistic invariant measures because this feature is frequently typical of them (see, for example, [5]). Then (13) can be written in the form

$$\left\|\left[Z_{t+1}\right]\right\| = \left|\frac{1}{\lambda} Z_{t+2} - Z_{t+1}\right| = \left(\frac{1}{\lambda} - 1\right)\left|Z_{t+1}\right| \tag{17}$$

with using the invariant with respect to unit shift.

It follows from (17) that for $|1/\lambda - 1| < 1$, i.e. for $\lambda > 1/2$, the right part of (17) converges to zero for $t \to \infty$, and, consequently, the variation λ^2 of martingal process Z_t converges to zero for $t \to \infty$. So, the statement about the convergence of the variation to zero is proved.

So on, we must prove that the embedding $X_t \to \int_0^1 X_t \mathrm{d}M$ is continuous. If it is true, we can simply approximate the process $Z_t = \int_0^1 X_t \mathrm{d}M$ by process X_t almost continuously. In the space $SL^1(M)$, it is not true in the common case, but such continuous embedding can be made, if X_t belongs to the class $SL^2(M)$, that is to the family of L^2-integrable λ^2-martingals.

Indeed, if it is true, then SL^2-continuous embedding of invariant measure μ_λ as the measure of random trajectories of map f_λ on the some measure γ_μ, which is constructed on the state of martingal M with respect to integration, is true.

This statement has been proved in [10] (in some different language) by the methods of nonstandard analysis and immediately following possibility of prolongation of the statement of the theorem of Romanenko–Sharkovsky about stochastic ordering with respect to absolutely continuous invariant measure μ_λ on the nonstandard universum $^*R$ on the norm

$$\gamma_\mu(\omega,t) = M(\omega,t)P\omega. \tag{18}$$

for some λ^2-martingal μ, where the value $\gamma_\mu(\Omega \times T) = E([M](1))$ is finite. $[M]$ denotes the variation of M.

For example, for Andersonian evolution B, we have $\gamma_B = P \times \lambda$, where λ is a normed «counting» measure on T, which is (in some defined sense) identical to known counting measure $\hat{\mu}_\lambda$ which was constructed by Misiurewicz [11].

Then the above-mentioned notation simply means that the process $Z_t = X_t \mathrm{d}\mu_t$ converges with respect to measure $\gamma_m(\omega,t)$ in L^1 in the nonstandard universum to some asymptotically periodic piecewise-constant function in average

$$E^Z f(Z_t) = \int S(t) f \mathrm{d}\gamma \tag{19}$$

and process X_t converges with respect to measure to a periodic function (it follows from the known results for difference equations with continuous time [11]):

$$E^X f(X_t) = \int s(t) f \mathrm{d}\mu \tag{20}$$

where γ is the Dolean's measure, and μ is the Jakobson–Misiurewicz measure [11]. In this case the point EZZ_t may be called fixed in average, if the equality $E^Z f(Z_t) = EZ_t$ is valid.

If $X_t \in SL^2$, the nonstandard version of the theorem of Romanenko and Sharkovsky is simply «the continuous embedding» of the standard version of this classical theorem.

We explain it in details. Let $f: I \to I$ be any unimodal map, i.e. the map which has one extremum. I is some open closed interval and $f \in C^2(I,I)$. We note by $A_n = A_n(f)$ the set of cycles of period n of map f which contains a point of extremum (maximum) c, so that $A_n(f) \neq \varnothing$, where $\varnothing$ is an empty set. Then the set A_n contains the maximal element (with respect to embedding), i.e. an indexed set $A_n^{(\alpha)}$ is bounded from above by the cycle of interval $A_n^{(\beta)}$, if the embedding $J_i^{(\alpha)} \subset J_i^{(\beta)}$ for each $i = \overline{0, n-1}$ is true [1]. The set $F = \left\{A_n^{(\alpha)}\right\}$, $\alpha \in G$ is ordering in the defined sense and elements of F are bounded from above by the cycle of intervals $A_n = \left\{\overline{\bigcup_{\alpha\in G} J_0^{(\alpha)}}, \ldots, \overline{\bigcup_{\alpha\in G} J_{n-1}^{(\alpha)}}\right\}$, where «the line» means the cloud of sets. By Zorn's lemma, the partially ordering set A_n contains the maximal element $A_n^* = \left\{J_{n,0}^*, \ldots, J_{k,k-1}^*\right\}$, and it can be stated that the extremum point C and, consequently, the cycle of interval A_n^* is defined «one-to-one».

Let $\{p_m\}_{m=1}^{m^*}$ be the increasing sequences of natural numbers such that $A_{p_m}(f) \neq 0$, $m < \infty$. We define $\varphi_m^* = \left\{x \in J : J \in A_{p_m}^*\right\}$. Then $f\left(\Phi_m^*\right) \subset \Phi_m^*$ and sequences $\left\{\Phi_m^*\right\}_{m=1}^{m^*}$ cab be constructed being familiar to filtration. So it is possible to triangle the set of all trajectories on the set of some classes in the sense of its classification with respect to ω-limit points. The «field of trajectories» $f\left(\xi_t\right)$ can be resembled, where ξ_t is the random process. In other words, the ordered stochastic filtration is constructed with respect to a probabilistic measure μ_λ (not Lebesgue measure). As will be proved below, such invariant measure exists.

Then it follows from theorem 6.6 [9] that for map f for $p_{m+1} / p_m = 2$ and $m^* = \infty$ for each $m < m^*$, there exists a unique invariant probabilistic measure μ of all periodic trajectories which is equal to zero on the closing Per(f) of set on the each subset of the set Per(f). For each continuous point $y \in P_0\left(C_\infty^{(0)}(f)\right)$, the equality can be written

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g\left(f^k(y)\right) = \int f \mathrm{d}\mu \qquad (21)$$

where $C_m^0 = \Omega_m^0 \bigcap \mathrm{Per}(f)$, and equality (21) is usual statement of Birggoff–Hinchin theorem in the discrete case. Here $P_0(f)$ is an invariant subset of set Per(f). Intuitively, as known, this theorem simply states that at any neighborhood of the set $\overline{\mathrm{Per}(f)}$, when the trajectories belong to such neighborhood, the frequency of the «coordinates» may be calculated by formula (21) in the «averaging sense», that is with respect to some measure μ.

The problem is how to construct such measure. Familiar candidates in statistical mechanic are so-called «δ»-masses of Dirac which describe the probability of observing the particle in some neighborhood of the given trajectory of its dynamics (see, for example, [8]).

If there is self-stochasticity here (i.e. if for deterministic map, the measure exists which is constructed by Jakobson-attracting periodic orbits), we define $\lambda > 1$, not $\lambda > 1/2$. There are simply «fixed orbits» which are invariant with respect to some shifting operator. Then by the theorem 6.7 [9], map f (more exactly, random trajectories of this map) has an invariant measure which is absolutely continuous with respect to Lebesgue measure.

In some sense,

$$\int X_t \mathrm{d}\mu_t = \int f\left(X_t\right) \mathrm{d}\mu_t \qquad (22)$$

for $t \to \infty$ on $\overline{\mathrm{Per}(f)}$, where $X_t \in O_{\beta \overline{\mathrm{Per}(f)}}$ small enough $\beta > 0$, i.e. in some neighborhood of a periodic random trajectory. It was shown above that in the space SL^2, embedding $X_t \to \int X_t \mathrm{d}\mu_t$ is true. But if (22) is true, then embedding

$X_t \to \int X_t \mathrm{d}\mu_t$ is true too. Such embedding is indeed true, if $\mathrm{d}\mu(X_t) = p(X_t)\mathrm{d}X_t$, where $p(X_t)$ is the derivation of Radon–Nikodim, which is existing and continuous, as known. Then the continuous embedding has place in $SL^2(\mathrm{d}X_t)$ with respect to Lebesgue measure dx. Consequently, in the proof of the relations of (22) it provides the existing of the «filtrations» for the deterministic map.

Indeed, the problem is reduced to the approximation of the Jakobson measure $\mathrm{d}\mu_t$ by the martingal Dolean measures $\mathrm{d}\mu_t$ (in the sense of continuous embedding). But the first one is an approximation by the Lebesgue measures, i.e. in the conditions of the theorem 6.6 [9], embedding $C_\infty^{(0)} \subset \bigcap_{m>1} \Phi_m$ is true. So, the «bed» trajectories do not go far from the cycle of interval, where $\Phi_m^* = \bigcup_{i=0}^{p_{m-1}} J_m^{(j)}$, and intervals $J_m{}^0, \cdots, J_m{}^{p_{m-1}}$ construct the cycle of intervals of period $p_m = 2^m$. As shown by Misiurewicz [11], every interval can be associated with measure which is equal to 2^{-m} (in some specified sense, the measure of Jakobson for which one for $\lambda \in \Lambda$ is related to an invariant measure μ_t).

On the other hand, $\int X_t \mathrm{d}\mu_t$ with respect to martingal is true, i.e. with respect to Dolean measure γ_μ. Second, it is evident that the Misiurevicz measure is embedding on the λ-counting measure on giperfinite interval. So on, Dolean measure for λ^2-martingal can be constructed. A consequence is relation (22) in the sense of almost continuous embedding. So on embedding from the left in SL^2 does not mean a possibility of one-to-one correspondence (the related «stochastic filtration in nonstandard sense»). Below we present an example how to apply our situation to a random Andersonian process.

As shown in [10, p. 133], in the space of state Ω, i.e. on the set of intrinsic function ω: $T \to -1, 1$, where T is a hyperfinite interval, an Andersonian process B: $\Omega T \to {}^*R$ is defined by formula $B(\omega, t) = \sum_0^1 \omega(s)\sqrt{\Delta t}$, where $\omega(s) = X(\omega,s)$ defined by the stochastic integral

$$\int_0^t \chi \mathrm{d}B(\omega) = \sum_0^t \omega(s)\sqrt{\Delta t} = \sum_0^t \sqrt{\Delta t} = \frac{t}{\sqrt{\Delta t}}. \tag{23}$$

The integral evidently, converges to infinity for each finite small t. But this integral will be finite, if for $t \in T$ we define $\omega \in \Omega$ from the example so, that $\omega \uparrow t = \langle \omega(s) | s\langle t \rangle$, and consider the hyperfinite process $X_t : \Omega \times T \to R^*$ such, that from $\omega \uparrow t - \omega \uparrow t$ it follows that $X(\omega,\mathrm{t}) = X(\omega,t)$. Let A_t be an intrinsic algebra of subsets of set Ω which is defined by the sets of the form

$$[\omega]_t = \omega' \in \Omega \big| \omega' \uparrow t = \omega \uparrow t \tag{24}$$

where $[\omega]_t$ is the variation of ω.

We have shown above that the variation $[\omega]_t$ converges to zero for $t \to \infty$ almost everywhere (except the set of points such that its measure μ equals to zero: in partly, it may be Lebesgue measure). Then, this statement is true asymptotically in some cases for the relation (24). There exist A_∞-intrinsic algebra (analogy of filtration); there exists process X_t on such algebra of sets: it follows from [10, p. 134] and it is equivalent to the converging $\int_0^1 X_t \mathrm{d}B_t$, if and only if the random value $X_t(\bullet, t)$ is A_t-measurable for every t.

It is evident, if there is self-stochasticity (by Sharkovsky measure and only!) in the case of existing measure of Jakobson [2], there also exists a set of «cylindrical» functions

$$\Phi^*_{m,t} = X_t \in J : J \in A^*_{p^*_m}. \tag{25}$$

So on, something similar to $A^*_{p_m}$-algebra (or filtration) should be constructed because we deal with the random process with the invariant measure μ_λ for some deterministic map f_λ (in the sense of Jakobson). It may be the Markovian process. And consequently, some σ_{st}-algebra must exist which is defined by the sets of $\Phi_{m,t}*$.

In the same way, the standard extending of p is the state of the filtration $(\Omega,\{Bt\}t \in [0,1],L(P))$, which is defined by the filtration $(\Omega,\{A_t\},P)$. For every $t \in [0,1]$ we define

$$B_t = \sum{}_{kst}\left(\bigcup_{s\simeq t} L(A_s) \cup N\right) \tag{26}$$

where Σ_{kst} means that we define Σ-algebra determined by the sets $\bigcup_{s\simeq t} L(A_s) \cup N$, where $\cong$ is defined in the nonstandard (hyperfinite) sense in the extending universum $^*R$, and N is the class of zero-measure with respect to $L(p)$-measure. Here $L(P)$ is so-called Leab measure. It is clear that it is a *-prolongation of some countably additive measure in the extending universum $^*R$.

Following to [10, p .84], we cite an example which is exactly determined with respect to the technique used in the ergodic theory of one-dimensional maps [9]. We consider 2^N-space of sequences of 0 and 1, i.e. the space with measure $(2^N, B, \mu)$ defined by the following construction: for every multiproduct 2 = {0,1}, we define Σ-algebra of all subsets and «counting» measure which is determined by equal weight of any point of space. It is useful to interpret it with the measure defined by Misiurewicz; see theorem 6.6 [9]. Then B is an usual algebra in the product of measurable spaces, and μ is the measure that is the product of measure on multi-product.

Nonstandard method of modelling of this state in the choice of some $\eta \in {}^*N - N$, where N is the set of the natural numbers, and considering f the space $\Omega = \{0,1\}^\eta$ of all «intrinsic» consequences of zero and unity of length η. Let A be an algebra of all intrinsic subsets of set Ω. Intuitively, we determine more exactly the structure of approximation of trajectories to some «random attractor», because in this case we can see different velocity of converging of trajectories with respect to different infinitely small values. Then, for each $A \subset A$

$$P(A) = \frac{|A|}{2^{\eta}}. \tag{27}$$

where $|A|$ is intrinsic power (massiveness) of set $A \subset \Omega$, i.e. the number of elements in A, and $2^{\eta} = |\Omega|$.

Similar well-known construction is connected with the space $(2^N, B, \mu)$. Let be $st_{\eta} : \Omega \to 2^N$ is the map of bounding. Then for every $A \subset b$, the statement

$$\mu(A) = L(p)\left(st_{\eta}^{-1}(A)\right) \tag{28}$$

is true almost everywhere, with corresponding σ-algebras (the details of such relation see in [10, p. 85]. So on «one-to-one» mapping between σ_{st}-algebra in standard sense and σ_{st}-algebra in nonstandard sense can be constructed.

Application of the nonstandard method to the classical one-dimensional systems

We show how the known theorem 6.6 [9] can be formulated in the hyperfinite sense in this case. Let be $K = \{0, 1, \cdot\ldots, k-1\}$, $k \in N^* - N$. We consider Leab's space $(K, L(A), L(P))$, where A is the algebra of all subsets of the set K, and P is the counting measure. Let φ be an operator of shift on K defined by the formula

$$\varphi(t) = \begin{cases} t+1, & \text{if } t < k-1, \\ 0, & \text{if } t = k-1. \end{cases} \tag{29}$$

Then a dynamical system of the form (Ω, B, μ, T) is considered, where (Ω, B, μ) is a probabilistic (standard) space, and T is the transform such that it preserves the measure μ. Note that T is factor of «hypercycle» K in the sense that there is a measure-conserving transform: $g : K \to \Omega$ such that $g(\varphi(t)) = T(g(t))$ for $L(P)$ almost everywhere in K.

Letting $f = T$, we see that the statement (3) of theorem 6.6 [1] is true (with approximation to homeomorphism $f(\bullet) = g(\varphi(\bullet))g^{-1}$, where g^{-1} is an invertible map). And, analogically, all other statements for map $f(\bullet)$ of defined form with respect to Leab's measure $L(p)$ but not only for the probabilistic absolutely invariant with respect to Lebesgue measure dx are true.

Similar construction of the metric by Skorohod [1] is applied [3] to the formulation the theorem of Romanenko–Sharkovsky [6] about ω-limiting behavior of random trajectories for map f_{λ} with respect to some invariant probabilistic measure μ_{λ}.

One more remark is as follows. At last we say, that the map $X_t \to \int_0^1 X_t \mathrm{d}B_t$ considered as map from $L^2(L(p) \times m)$ to $L^2(L(P))$, where m is Lebesgue measure, conserves the norm:

$$E\left(\int_0^1 X(\omega,s)\mathrm{d}b(\omega,s)\right)^2 = E\left(\int_0^1 X^2(t)\mathrm{d}t\right). \quad (30)$$

Combined with nonequality of Shvarts [10, p. 82]:

$$E\left(|X|\right)^2 \le E\left(|X|^2\right), \quad (31)$$

(30) is reduced to prove the convergence of random trajectories to piecewise continuous periodic functions for the logistical map «to one side». Indeed, it is enough to consider the random process $Y_t^\lambda = \int_t^{t+1} X_S \mathrm{d}b_S$, instead of equivalent process X_t^λ, which has the probabilistic distribution with respect of Jakobson measure μ_λ. First of all, we should integrate each of two parts of terms of the logistical map about the random trajectories b_t and then use the operation of mathematical expectation. The nonequality of Hida is applied to «disordering» of the quadratic nonlinearity.

Note that this fact follows from the fundamental statement proved in [10] for the asymmetric simple exclusion process [12], of course, at a neighborhood of some invariant measure, also invariant by space translation τ, which is product measure γ^a such, that $\gamma^a(\eta(k)) = a$ for every k, with $a \in [0,1]$. So the equilibrium is on γ^a at level a or density a.

It is enough to prove that for any cylindrical function f on E with $\|f\| = \sup_{e\in E}|f(e)| < 1$, we have $E\left(If(X_t)\right) = E(I)E\left(f(X_t)\right)$, and f being fixed, we have the following property: for every $\varepsilon > 0$, there exist an integer N^ε such that for every function h in $C(E)$ with $\|h\| \le 1$ and $\mathrm{supp}(h) \subset \{0,1\}^{[-N^\varepsilon, N^\varepsilon]}$, the nonequality

$$\left|E\left(h(X_t)f(X_t)\right) - E\left(h(X_t)\right)E\left(f(X_t)\right)\right| < \varepsilon \quad (32)$$

is true, as follows from 1.4.6, part (c) of 1.3.9 in [7] and the fact, that the measure is a product one (for example, $\gamma^{a,b}$). And then, of cause, the inequality

$$\begin{aligned}&\left|h\left(X_t^2\right)f\left(X_t^1\right)\mathrm{d}\tilde{\gamma} - \int h\left(X_t^2\right|\mathrm{d}\tilde{\gamma}\int f\left(X_t^1\right)\mathrm{d}\tilde{\gamma}\right| = \\ &= \left|E\left(h_1(X_t)f(X_t)\right) - E\left(h_1(X_t)\right)E\left(f(X_t)\right)\right| < \varepsilon\end{aligned} \quad (33)$$

is true, since $h(X_t^2) = h_1(X_t^1)$ with $h_1 \in C(E)$, $\|h\| \le 1$ and $\mathrm{supp}(h) \subset \{0,1\}^{[-N^\varepsilon, N^\varepsilon]}$.

So, it is clear how to use this inequality to prove our statement in the vicinity of equilibrium. It is interesting that we do not verify the statement for all initial distributions for logistical equation (1), but only for the initial functions which belong (in the sense of distributions) to some of ε-neighborhood of invariant

Dolean measure. At that time, the theorem of Romanenko and Sharkovsky for almost all initial distributions on some massive set is true for Jakobson measure.

The considered problem can easily find application to a number of branches of physics. The existence of periodic low-frequency structures in the problem of variation of the properties of stochastic signal is known for dynamical systems (see, for example, [13], where the graphics of a computer experiment for «filtrational» stochastic trajectories is described). The methods can be applied to functional diagnostic of dynamical systems that is in wide use in science.

Applications to abstract problems of statistical physics

We recall that for given unimodal f (for example, defined by conditions of theorem 6.6 [9]), there exist the invariant measures γ_{μ_1}, also invariant by space translation τ, such that almost all (with respect to the measure) trajectories defined in hyperfinite sense are situated on the closing Per(f) in the sense of converging

$$\lim_{t\to\infty}\int_0^1 X_t^2 \mathrm{d}\gamma_\mu(t) \to \overline{\mathrm{Per}(f)} \tag{34}$$

where the convergence at the right part must be defined as «superweak» convergence, i.e. with respect to measure $\gamma_\mu(t)$ in the nonstandard sense and to measure μ in the standard one.

It is interesting (for example, from the viewpoint of computer modelling) how the limiting relation (34) in the «strong» sense can be defined. Here we consider only intuitive aspect of the method applicable for the problem solution. Indeed, as noted above (Lemma 1), in the case of self-stochasticity, if the map f_λ is not of attractive and non-hyperbolic orbit, then $f \in C^2(I,I)$ and has f_λ-invariant measure μ_λ absolutely continuous with respect to Lebesgue measure dx ($\mu_\lambda <$ dx).

From the Radon–Nikodim theorem, the probabilistic measure absolutely continuous with respect to a Lebesgue measure exists, if and only if, there is L^1 function such that

$$\mu(A) = \int_A P(x)\mathrm{d}x\,. \tag{35}$$

Note that for $SL^2(M_t)$ martingals, it may happen that such measure does not exist.

Suppose that δ(x)-masses of Dirac are situated on Per(f). This supposition is reasonable from the viewpoint of physics of many particles [4]. We derive from (34) and ergodic theorem of Birkhoff and Hinchin that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^1\int_0^1 X^2 \mathrm{d}\mu_t \to \int f\mathrm{d}\mu_t = \mathrm{Per}(f)\mu\,. \tag{36}$$

The theorem of Birkhoff and Hinchin is applicable here, because all t» measure and the Leab measure. So the iterated integral is defined correctly.

Let $\mu(x) = \lambda(x)\mathrm{d}x$. Then the last relation follows from (35). Usually, the analysis is «stopped» at (34) [5], because the deviation of Radon–Nicodim does not exist for the measure defined in this paper.

This remark is aimed to demonstration of the analogy to Romanenko and Sharkovsky theorem not only in the sense of weak convergence, but also «in each point»-convergence. Indeed, such statement (or supposition) can be easily derived from the Lemma for λ^2-martingals formulated above. It is sufficient to consider the random function

$$\mu_t = f(\eta_t) - \int_0^t \Omega f(\eta_s)\,\mathrm{d}s \qquad (37)$$

where **Ω** is Markovian generator, and for each $f \in D(\Omega)$ the function M_t is martingal ($D(\Omega)$ is the sensitive sphere of definition of the generator) [5].

Let $f(\eta) = \eta(x)$ at some fixed $x \in Z^1$. The average value $E^\eta|M_t|$ is bounded about t (theorem 3.4, [9]) and, consequently, by the theorem about of convergence of martingals almost everywhere, there exist the limit $\lim\limits_{t\to\infty} M_t$ and this limit is

$$\lim_{t\to\infty} \int_0^1 \Omega f(\eta_s)\,\mathrm{d}s = \int_0^\infty \Omega f(\eta_s)\,\mathrm{d}s \qquad (38)$$

almost everywhere (see theorem 3.4). Consequently, there exist $\lim\limits_{t\to\infty} f(\eta_t)$ almost everywhere (see [5, p. 334],). For $\lambda < \lambda_c^1$, the only possible value is zero, i.e. for each $\eta \in X$ and $x \in SP^\eta\,[\eta_t(x)] = 0$ for $t = 1$.

But it is easily seen, that for a complicated map f_λ we can define λ_c^2 that for $\lambda > \lambda_c^2$ the set of all invariant measures $\mu \in J$ is consisting of two points $\{J_1(\lambda), J_2(\lambda)\}$. As a result we receive an «analogy» of Romanenko and Sharkovsky theorem in the strong sense of convergence to random ω-limit set with probability $p^\eta[\eta_t(x) = \bullet] = 1$.

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«МАРТИНГАЛЬНОЕ УПОРЯДОЧЕНИЕ» ШАРКОВСКОГО ДЛЯ ПРОСТОГО ЛОГИСТИЧЕСКОГО ОТОБРАЖЕНИЯ

Представлены результаты анализа поведения однопараметрического семейства функций $f_\lambda(x)$, определяемого непрерывным разностным уравнением и имеющего периодическую орбиту-аттрактор. Показано, что все вероятностные f_λ-инвариантные меры являются сингулярными по отношению к мере Лебега dx, и итерации $f^n_{\lambda^*}\mathrm{d}x$ сходятся в слабой *-топологии к дискретной инвариантной мере. Предполагается, что данная ситуация является типичной с топологической точки зрения. Это утверждение доказано для асимптотического поведения случайных траекторий, если разрешены только переходы к ближайшему окружению. Классическим примером соответствующих физических проблем является мера Винера, но случайные процессы, как правило, являются Гауссовыми.

Ключевые слова: разностное уравнение, считающая мера, мартингал, фильтрация, аттрактор