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I.B. Krasnyuk, T.N. Melnik, V.M. Yurchenko

SURFACE-DIRECTED TWO-DIMENSIONAL WAVE STRUCTURES IN CONFINED BINARY MIXTURES

Donetsk Institute for Physics and Technology named after A.A. Galkin

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The process of formation of two-wave structures in binary mixtures (for instance, polymer blends or binary alloys), which is described by the Cahn–Hilliard (CH) equation with dynamic boundary conditions with feedback, is considered. Asymptotic periodic impulse structures of relaxation, pre-turbulent and turbulent type with finite, countable or uncountable number of fronts of discontinuities per a period are studied. It is found that asymptotic structures are elements of the attractor of the initial value boundary problem. Application to a square-shaped binary polymer is considered. Computer simulation describing the formation of crystallites in melts is done.

Keywords: Cahn–Hilliard equation, nonlinear dynamic boundary conditions, limit distributions of relaxation, pre-turbulent or turbulent type

Introduction

Last years, network scientists have directed their interest to the multi-layer character of real-world systems, and explicitly considered the structural and dynamical organization of graphs made of diverse layers between its constituents. Most complex systems include multiple subsystems and layers of connectivity and, in many cases, the interdependent components of systems interact through many different channels. Such a new perspective is indeed found to be the adequate representation for a wealth of features exhibited by networked systems in the real world. The contributions presented in this Focus Issue cover, from different points of view, the many achievements and still open questions in the field of multi-layer networks, such as: new frameworks and structures to represent and analyze heterogeneous complex systems, different aspects related to synchronization and centrality of complex networks, interplay between layers, and applications to logistic, biological, social, and technological fields.

Formulation of the problem

In this paper, we study an initial value boundary problem, which describes the process of formation of two-dimensional spatial-temporal structures in symmetric diblock copolymers, which are confined by unite cube. We consider evolution of one component of a binary mixture, which is modeled by the two-dimensional convective Cahn–Hilliard equation [1] with dynamic boundary conditions with feedback and some initial conditions. This model describes distribution of concentration in a binary melt after surface (super)cooling. We assume that velocities of nucleation of crystallites, which arise on the sides of the square, are nonlinear functions on concentration and temperature that correspond to real experimental data [2]. Indeed, the frequency of nucleation can be approximated by a parabolic function with negative second derivatives with respect to the temperature of crystallization ([2], Fig. 60). Hence, the flow of concentration $J \sim u_x$ is the same nonlinear function at the sides of the square $x = 0$ and $x = l$. The same statement is true for the coordinates $y = 0$ and $y = l$, where l is the size of the system. Then we can consider the boundary conditions

$$u_x(t,0,y) = F_1\left[u_x(t,0,y)\right], \quad u_x(t,l,y) = F_2\left[u_x(t,l,y)\right], \tag{1}$$

$$u_y(t,x,0) = G_1\left[u_y(t,x,0)\right], \quad u_y(t,x,l) = G_2\left[u_y(t,x,l)\right], \tag{2}$$

where F_1, F_2, G_1, G_2 are given non-linear functions, and the initial conditions must be added. In a similar way, we can consider boundary conditions for the derivatives u_{xxx} and u_{yyy}, respectively. Next, the CH-equation can be linearized in the vicinity of the disordered phase $u = 0$, and, as a result, we have the initial value boundary problem with non-linear boundary conditions.

The main observation is that solutions of the problem can be found as

$$u(t,x,y) = f(t + a_1 x) + g(t + a_2 y), \tag{3}$$

where f and g are unknown functions; a_1 and a_2 are parameters of the problem. Then it will be proved that the functions $f(\zeta)$ and $g(\eta)$ tend to periodic piecewise constant limit functions as $t \to +\infty$ with finite or infinite number of the points of discontinuities per periods. The resulting solutions are presented in Fig. 1.

Indeed, the problem can be reduced to the difference equation

$$f\left(t + \frac{l}{a_1}\right) = \Phi_1\left[f(t)\right] \tag{4}$$

for function f, and to a similar difference equation for function g, but with another map Φ_2. The solutions of these equations are asymptotic periodic piecewise constant impulses (see [3]). Hence, an asymptotic solution $u(t, x, y)$ is the sum of such impulses. In 1D-case, the problem has been solved for confined binary alloys [4] and for polymer blends [5].

Let's consider a binary mixture which is placed into a square of size l. Assume that the mixture L is supercooled to the temperature $T < T_g$, where T_g is the temperature of phase surface decomposition. We suppose also that we are dealing with a bulk copolymer melt which is in a disordered state $u = 0$ initially and $u(t, x, y)$ is one of the components of the binary mixture.

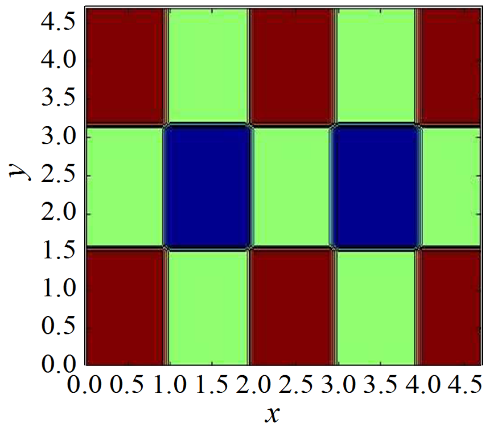

Fig.1. Limit distributions of relaxation type in 2D-case

Next, we consider the convective CH-equation of the form [1]:

$$u_t - a_1 u_x - a_2 u_y = \left[k_1 u\left(1-u^2\right)+u_{xx}\right]_{xx} + \left[k_2 u\left(1-u^2\right)+u_{yy}\right]_{yy} \qquad (5)$$

with the boundary conditions [5,6,7]:

$$u_x = F_1[u] \text{ at } x=0, \quad u_x = F_2[u] \text{ at } x=l, \qquad (6)$$

$$u_y = G_1[u] \text{ at } y=0, \quad u_y = G_2[u] \text{ at } y=l, \qquad (7)$$

$$u_{xxx} = \Upsilon_1[u] \text{ at } x=0, \quad u_t = \Upsilon_2[u] \text{ at } x=l, \qquad (8)$$

$$u_{yyy} = \Psi_1[u] \text{ at } y=0, \quad u_t = \Psi_2[u] \text{ at } y=l, \qquad (9)$$

where F_k, G_k, Υ_k, $\Psi_k \in C^4(I,I)$ are given functions, and I is an open bounded interval. The initial conditions are

$$u(x,0) = u_0(x). \qquad (10)$$

If the right-hand sides of the boundary conditions are zero, then we obtain the double Neumann boundary conditions. We can also consider so-called dynamic boundary conditions with additional terms u_{ttt} and u_t. Indeed, as remarked in [6], «a dynamic boundary condition recently proposed by some physicist to account for interactions with the walls». Further, as noted in [8,9,10], dynamical interaction with the boundary surface must be taken into account for some materials. Mathematically, this fact corresponds to the considering a free energy functional of a system, which contains also a boundary contributions.

Thus, we can consider classical the double-Neumann or double-Robin type boundary conditions or non-classical boundary conditions of Wentzell type (see [6]). It means that the surface rate or the flow of concentration is proportional to the variational derivative $\delta F[u]/\delta u$, where $F[u]$ is the free energy functional on the surface Ω:

$$\tau_u = -\gamma\delta F[u]/\delta u\,, \tag{11}$$

where τ_u is the relaxation time of the concentration of binary mixture, γ is proportional to the frequency of collision between the atoms on the flat wall which confines the binary mixture. The right side of (11) plays the role of the driving force. In particular, we can choose

$$\delta F[u]/\delta u := \alpha\,\Delta_\Gamma u - \partial_n u - \beta u\,. \tag{12}$$

Here α, β are constants, $\Delta_\Gamma u$ is the Laplace–Beltrami operator. Without this operator, we obtain the classical Robin boundary conditions

$$\tau_u u_t = -u_x u - \beta u \tag{13}$$

in the one-dimensional case on the flat wall. At first, being applied to polymer binary mixtures, this type of the boundary condition has been considered in [11,12]. For binary alloys, such conditions have been considered in [5] for the first time.

The present paper is aimed to considering of generalized boundary conditions in the form

$$\tau_u u_x = \Phi\left[\partial_n u,\, \partial_n^3 u,\, u\right], \tag{14}$$

where Φ is a nonlinear function with feedback that depends also on additional parameters of the problem. We confined ourselves by the study of the convective CH-equation in a bounded domain $\Omega \subset R^3$. For simplicity, we assume that Ω is a cube of size l. On each surface in x, y-space, there are dynamic boundary conditions of type (14) with additional «classical» stationary boundary conditions

$$\partial_n^3 u = \Psi\left[\partial_n u,\, \partial_n^3 u,\, u\right] \tag{15}$$

on $\partial\Omega$, where $\Psi[\cdot]$ is a given functional. Indeed, the homogeneous boundary condition for the CH-equation is well-known in literature

$$\partial_n^3 u = 0\,. \tag{16}$$

Reduction of the problem to difference equations of continuous type

We begin from the determination of function $u = (N_A - N_B)/(N_A + N_B)$ called an order parameter, and where the chain of N_A subunits is covalently bonded to the chain of N_B subunits, and $N_A + N_B = 1$; a^{-1} is a dimensionless velocity of convection. Here, $k := k - k_c$ is the Flory–Huggins parameter, k_c is the critical value of phase decomposition. Note that if $k > k_c$, there are sinusoidal fluctuations in the vicinity of the disordered phase $u = 0$. If $k < k_c$, there are monotone fluctuations of the order parameter.

Further, at a neighborhood of the mean value $u = 0$ (when $N_A = N_B$), we can consider the linearized CH-equation:

$$u_t + a_1 u_x + a_2 u_y = \left[k_1 u + u_{xx}\right]_{xx} + \left[k_2 u + u_{yy}\right]_{yy}. \tag{17}$$

A solution of this equation can be found in the form

$$u(t, x, y) = f\left(t - x/a_1\right) + g\left(t - y/a_2\right). \tag{18}$$

We assume that a_1, $a_2 > 0$. Then, substituting representation (18) into equation (17), we obtain two ordinary difference equations:

$$\lambda_1 f_{\zeta\zeta}(\zeta) + f_{\zeta\zeta\zeta\zeta}(\zeta) = 0, \tag{19}$$

$$\lambda_2 g_{\eta\eta}(\eta) + g_{\eta\eta\eta\eta}(\eta) = 0, \tag{20}$$

where $\zeta = t + x/a_1$, $\eta = t - y/a_2$ and $\lambda_1 = k_1 a_1^2$, $\lambda_2 = k_2 a_2^2$.

We assume that $\lambda_1 < 0$. By definition, $\lambda_1 = \chi - \chi_c$, where χ is the Flory–Huggins parameter, which characterizes the interaction between atoms in polymer blends [13]. That is $\chi < \chi_c$, where χ_c is the critical parameter of phase decomposition of the disordered phase of polymer melt into two ordered phases. It will be shown that if $\chi - \chi_c$, then there are monotone distributions of the concentration of one component of a binary mixture. In this case, surface perturbations are dominating, and we can speak about surface induced ordering in the melt volume, as $T < T_c$, where T_c is the critical temperature of surface decomposition into two ordered surface-induced phases.

Then we can find solutions exactly. Indeed, we assume that a_1, $a_2 > 0$. Then it follows from (18) that function $f(x(t),t)$ constant along characteristic $\mathrm{d}x(t)/\mathrm{d}t = a_1$, function $g(y(t),t)$ is constant along characteristic $\mathrm{d}y(t)/\mathrm{d}t = a_2$. Next, after integrating equations (19), (20) from points $\zeta_0 = t$, $\eta_0 = t$ to points $\zeta_0 = t + 1/a_1$ along characteristics $\mathrm{d}x(t)/\mathrm{d}t = a_1$ and $\mathrm{d}y(t)/\mathrm{d}t = a_2$, respectively, we have the following relation:

$$f'(l, t) - f'(0, t) = C_1 \exp(\alpha_1 t) + C_2 \exp(\alpha_2 t), \tag{21}$$

where $\alpha_1 = \sqrt{\lambda_1}$, $\alpha_2 = \sqrt{\lambda_2}$ and C_1, $C_2 \in R$. Then it follows from relations

$$f(l, 0) - f(0, 0) = C_1 \alpha_1^{-1} + C_2 \alpha_2^{-1} + C_3, \tag{22}$$

$$f'(l, 0) - f'(0, 0) = C_1 + C_2, \tag{23}$$

$$f''(l, 0) - f''(0, 0) = C_1 \alpha_1 - C_2 \alpha_2, \tag{24}$$

that

$$C_1 = \left(\alpha_1 + \alpha_2\right)^{-1} + \alpha_2 \left[f'(l, 0) - f'(0, 0)\right] + f''(l, 0) - f''(0, 0), \tag{25}$$

$$C_2 = \alpha_2^{-1} \left[C_1 \alpha_1 - f''(l, 0) + f''(0, 0)\right], \tag{26}$$

$$C_3 = f(l, 0) - f(0, 0) - C_1 \alpha_1^{-1} - C_2 \alpha_2^{-1}. \tag{27}$$

We confined ourselves by the study of the case of $C_1 = 0$. Then

$$f'(l,t) - f'(0,t) = C_2 \exp(\alpha_2 t), \tag{28}$$

Where $\alpha_2 < 0$. We derive from (28) that

$$f'(l,t) - f'(0,t) \Rightarrow 0 \tag{29}$$

as $t \to +\infty$ for all points $t \in R^+$.

Next, after integrating (19) from $\zeta = t$ $\zeta = t$ to $\zeta = t - l/a_1$, we obtain:

$$f'''(t) - f'''(t - l/a_1) + \lambda_1 [f'(t) - f'(t - l/a_1)] = 0. \tag{30}$$

Now we use double Neumann–Neumann boundary conditions. Then a functional equation follows from differential-difference equation (30):

$$G_1[f(t)] - F_2[f(t - l/a_1)] + \lambda_1 \{G_1[f(t)] - F_2[f(t - l/a_1)]\} = 0. \tag{31}$$

Further we suppose that equation (31) can be solved so that

$$f(t) = \Phi_1[f(t - l/a_1)], \tag{32}$$

where $-l/a_1 \geq t < 0$. Map $\Phi_1 \in C^2(I, I)$ where I is an open bounded interval. We consider only the class of so-called unimodal maps [3], which have only one extreme point. An example is the well-known logistic map $u \to \lambda u(1-u)$, which maps interval [0,1] into itself.

In the same way, we can obtain the difference equation for function $g(\eta)$

$$g(t) = \Phi_2[g(t - l/a_2)], \tag{33}$$

where $\alpha_2 > 0$. The solution of equation (32) can be found with step by step iterating of the initial function $h_1(t)$ on $-l/a_1 \leq t < 0$ with the aid of map $\Phi_1 : I \to I$. In typical cases, limit solution $p_1(t)$ is a piecewise constant periodic function with finite, countable or uncountable set of points of discontinuities Γ on a period. If Γ is finite, then we deal with solutions of relaxation type. If Γ is countable, then we have limit distributions of pre-turbulent type. If Γ is uncountable, then we have limit distributions of turbulent type (see Fig. 1).

Here, $p_1(t) \in P^+$ for almost all points $t \in R^+$ excluding a set of points of meager Lebeque zero, where P^+ is a set of attractive fixed points of map Φ_1 [3]. Parameter $\lambda_1 = k_1 a_1^2$ is a parameter of period doubling bifurcations of limit solutions because map Φ_1 depends on this parameter. Thus, the mobility of atoms k_1 and the velocity of convection a_1 are bifurcation parameters. Note that the mobility $k_1 = = k_1(\theta)$ can be a monotone function depending on dimensionless temperature θ. But there is also a special case $k_1 :\sim k_1\theta^{-1}$. If a mixture is cooled at flat walls, then oscillating structures with non-monotone amplitudes appear at the surface. Next, these wave-type structures propagate into the volume of the mixture with exponential decay amplitudes. If $t \to +\infty$, then the spatial temporal structures tend to the function

$$p(\zeta,\eta)=p_1(\zeta)+p_2(\eta), \tag{34}$$

where $\zeta=t-x/a_1$ and $\eta=t-y/a_2$. Here $p_1(\zeta)\in P_1^+$ and $p_1(\zeta)\in P_2^+$, where P_1^+, P_2^+ are attractive fixed points of maps Φ_1 and Φ_2, respectively. In Fig. 2, the results of computer simulation of limit distributions of concentration are presented.

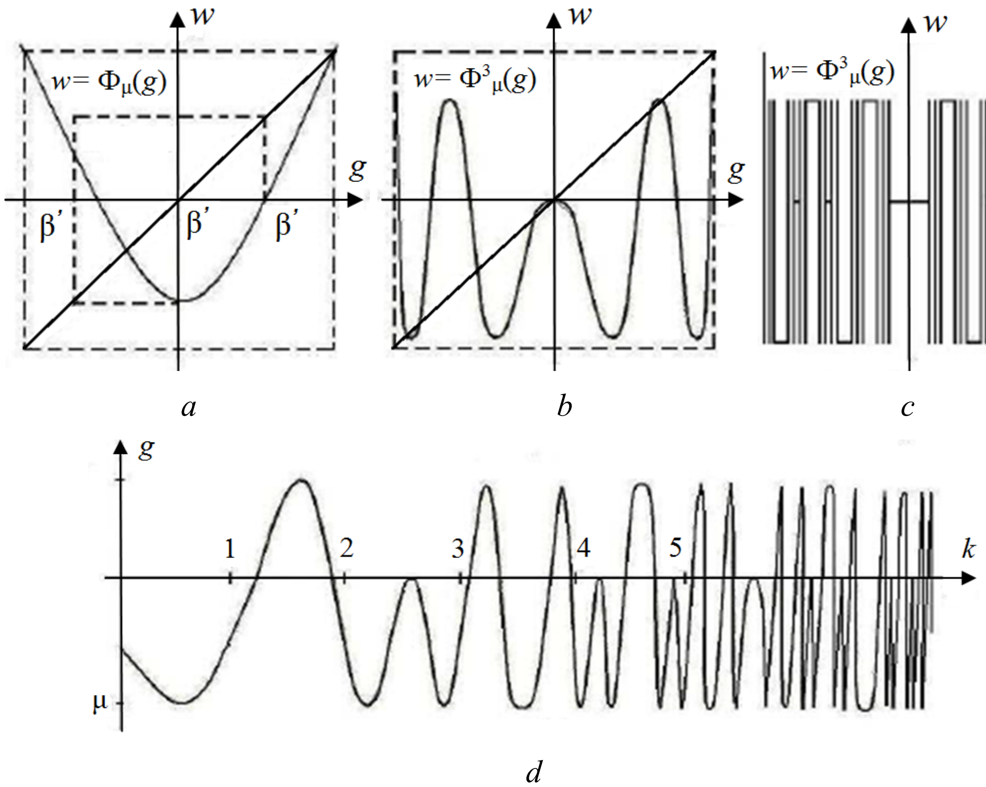

Fig. 2. Limit distributions of pre-turbulent type with countable points of discontinuities per a period

For limit distributions of pre-turbulent type, there are thickening lines. For limit distributions of turbulent type, there are Cantor type sets of lines of discontinuous for wave fronts of propagation of concentration in the volume. It is similar to well-known Sierpinski carpet. As a result, there is directional ordering, so that in *Ox* direction we have solutions of relaxation type for example, and solutions of relaxation type in *Oy* direction. There are also the following cases: relaxation $\oplus$ pre-turbulent type, relaxation $\oplus$ turbulent type, relaxation $\oplus$ turbulent type, pre-turbulent $\oplus$ turbulent type and so on.

Reduction of problem to a system of difference equations at a plane

In this section, we consider common case without surface solidification of the melt one in one direction. Then the problem can be reduced to the functional relations:

$$G_1\left[f(t)+g(t-y/a)\right]-G_2\left[f(t-l/a)+g(t-y/a)\right]+$$

$$+\,\lambda\Big(F_1\big[f(t)+g(t-y/a)\big]-G_2\big[f(t-l/a)+g(t-y/a)\big]\Big)=0\,, \qquad (35)$$

$$\hat{G}_1\big[f(t-x/a)+g(t)\big]-\hat{G}_2\big[f(t-x/a)+g(t-l/a)\big]\;+$$

$$+\,\lambda\Big(\hat{F}_1\big[f(t-x/a)+g(t)\big]-\hat{F}_2\big[f(t-x/a)+g(t-l/a)\big]\Big)=0\,, \qquad (36)$$

where we assume that $l_1 = l_2 = l$ and $a_1 = a_2 = a$. Since functions $f(t-x/a)$ and $g(t-y/a)$ are constant along lines $\mathrm{d}x/\mathrm{d}t = a$ and $\mathrm{d}y/\mathrm{d}t = a$, we can consider relations (35), (36) only at points $x = l$ and $y = l$. So we obtain that

$$G_1\big[f(t)+g(t-l/a)\big]-G_2\big[f(t-l/a)+g(t-l/a)\big]\;+$$

$$+\,\lambda\Big(F_1\big[f(t)+g(t-l/a)\big]-G_2\big[f(t-l/a)+g(t-l/a)\big]\Big)=0\,, \qquad (37)$$

$$\hat{G}_1\big[f(t-l/a)+g(t)\big]-\hat{G}_2\big[f(t-l/a)+g(t-l/a)\big]\;+$$

$$+\,\lambda\Big(\hat{F}_1\big[f(t-l/a)+g(t)\big]-\hat{F}_2\big[f(t-l/a)+g(t-l/a)\big]\Big)=0\,. \qquad (38)$$

Next, we assume that these relations are solvable so that

$$f(t)=\Phi_1\big[f(t-l/a),\, g(t-l/a)\big], \qquad (39)$$

$$g(t)=\Phi_2\big[f(t-l/a),\, g(t-l/a)\big]. \qquad (40)$$

If the map $\Phi:(f,g)\to\big[\Phi_1(f,g),\Phi_2(f,g)\big]$ is a structurally stable hyperbolic map, then functions $f(t)$, $g(t)$ are asymptotic $2^N l/2$-periodic piecewise-constant functions with finite or infinite number of the points of discontinuities per a period, where N is the least common multiple of attractive circles of the map $\Phi: I\times I\to I\times I$ [14].

Points of discontinuities

Let us consider an initial curve $\nu_h=\{u\in R^n : u=h(t),\, t\in[0,1)\}$ and define the set $H^+=\{h(t)\in H\}$ such that: (i) $\nu_h\cap W^s(a)=\varnothing$ if $\dim W^s(a)<n-1$; (ii) if $h(t')\in W^s(a)$ and $\dim W^s(a)<n-1$. Then at $t=t'$ the curve ν_h intersects $W^s(a)$; (iii) $\det D\varphi_u\neq 0$ if $u\in W^s(a)$ and $\dim W^s(a)=n-1$; (iiii) if unstable manifold $W^u(a)$ of point $a'\in\Omega(\varphi)$ such that $\dim W^u(a)<1$ intersects with a stable manifold $W^s(a')$ of point $a'\in\Omega(\varphi)$, then this intersection is transversal.

Then we obtain from (iiii) that the map φ^N has a one-dimensional separatrix going from one saddle to another saddle. H^+ is a set of the second category in C^0 topology, at follows from 1D-case [3]. Function $p_h(t)$ is multivalued on the set

$$\Gamma_p=\bigcup_{i=0}^{\infty}\Big\{t\in[i,i+1): h(t-i)\in W^s(a),\; a\in P^{\pm}\Big\}, \qquad (41)$$

$p(t) \in C^0\left(R^+\Gamma_p, P^+\right)$ and $p(t) \equiv \varphi^{i \bmod N}(a) \equiv p^*(t)$ on every interval $J \in [i, i+1)$, $i = 0, 1, \ldots$ such that $J = J'(\mathrm{mod}\,1)$, $h(J') \subset W^s(a)$, $a \in P^+$.

Thus, three-dimensional CH-equation with nonlinear flow at the facet of the cube and the nonlinear dynamic boundary conditions has been considered. It is shown that the IBVP has a unique oscillating solution with non-decreasing amplitude which tends to a piecewise constant periodic distributions in the form of $u(x, y, z, t) = P(t + \lambda_1 x, t + \lambda_2 y, t + \lambda_3 z)$ as $t \to +\infty$, where $\lambda_k \in R$, $k = 1, 2, 3$. Distribution $P(\zeta_1, \zeta_2, \zeta_3)$ is $2^N p$-periodic on every argument; $p = \lambda_k n_k$, $k = 1, 2, 3, \ldots, n_k = 2, 3, \ldots$, and N is the least common multiple of the periods of attracting cycles of map $\Phi : R^3 \to R^3$ where $\Phi = \left(\Phi_1^{n_1}, \Phi_2^{n_2}, \Phi_3^{n_3}\right)$. As a result, the problem is reduced to the equations:

$$\hat{u}_k(t + p) = \Phi_k\left[\hat{u}_1(t), \hat{u}_2(t), \hat{u}_3(t)\right], \ \Phi_k := \Phi_k^{n_k}, \ k = 1, 2, 3, \ u_k := \Phi_k^{n_k}\left[u_k\right], \quad (42)$$

where u_k are the original components of solutions. As noted above, asymptotic behavior of the solution of system (42) is known.

Applications to polymer systems. 2D-case

We assume that polymer blends are in a disordered state $u = 0$. Then under action of surface (super)cooling, blends form ordered regions of 1 micrometer by the order. We consider spatial-temporal ordering in a melt which arise for surface temperatures $T_g < T < T_m$, where T_g is the glass temperature, and T_m is the melting temperature. We assume also that there is surface cooling with feedback which results in nucleation of crystallites of nanometer-size areas. Such process of nucleation will be described by dynamic boundary conditions with feedback.

We consider a binary mixture which is placed in a square of size l. It will be shown that surface nucleation with feedback results in appearing spatial-temporal traveling waves of relaxation, pre-turbulent and turbulent type in the square. In Fig. 2, the results of computer simulation are shown. Similar distributions are typical for nanometer-side crystallites for Au or NaCl. An example of latter case is the change in color of Cd crystals [15].

We also show mathematically what a scenario of change of the dimensionality of the system is. Indeed, as noted in [15], Fig. 6: «If a NsM consists of thin needle-shaped or flat, two-dimensional crystallites, only two or one dimension of the building blocks becomes comparable with the length scale of the physical phenomenon». In these case, the NsM becomes a two-or or dimensional system according to this phenomenon.

The ordering effect can be described mathematically by composition of limit solutions of the form: limit constant solutions $\oplus$ oscillating solutions of relaxation type. For decomposed waves $f(t - x/a)$ and $g(t - y/a)$, it means that in Ox-direction the boundary conditions are

$$u_t = F_1[u]\big|_{x=0}, \quad u_t = F_2[u]\big|_{x=l} \tag{43}$$

and in Oy-direction the boundary conditions are

$$u_t = G_1[u]\big|_{y=0}, \quad u_t = G_2[u]\big|_{y=l}, \tag{44}$$

where F_1, F_2 are monotone functions, and G_1, G_2 are non-monotone functions. There are also situation when we have limit distributions constant $\oplus$ constant, constant $\oplus$ pre-turbulent, constant $\oplus$ turbulent, relaxation $\oplus$ relaxation, relaxation $\oplus$ pre-turbulent, relaxation $\oplus$ turbulent, pre-turbulent $\oplus$ turbulent and turbulent $\oplus$ turbulent types in Ox- and Oy-directions, respectively.

Conclusions

In this paper, theoretical results describing different scenarios of the surface induced ordering in a confined 2D-binary mixture have been considered. A new type of «mixed multi-directed turbulence» is constructed mathematically, so that in every 2D = 1D $\oplus$ 1D-direction we have a different type of «turbulence». Thus, there are limit distributions in a square with dynamic non-linear nucleation of crystallite from the sides of square, which results in oscillating piecewise-constant periodic distributions of concentration of binary mixture with finite, countable or uncountable number of fronts of discontinuities per a period. The number of oscillations in every 1D-direction can be different. Thus we have different oscillating behavior of limit distributions of concentration in different directions of the sides of the square. Mathematical results have been compared with the experiment.

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1. *A. Eden, V.K. Kalantarov*, Appl. Math. Lett. **20**, 455 (2007).
2. *V.P. Skripoff, V.P. Koverda*, Spontaneous crystallization of supercooled liquid, Nauka, Moscow (1984) [In Russian].
3. *A.N. Sharkovsky, Yu.L. Maistrenko, E.Yu. Romanenko*, Difference Equations and Their Applications, Ser. Mathematics and Its Applications, Kluwer Academic, Dordrecht (1993).
4. *I.B. Krasnyuk*, Int. J. Comput. Mater. Sci. Eng. **2**, 1350006 (2013).
5. *I.B. Krasnyuk, R.M. Taranets, M. Chugunova*, Physica **A415**, 19 (2014).
6. *C.G. Gal, M. Grasselli*, Discrete Contin Dyn Syst 22, 1009 (2008).
7. *S. Puri, K. Binder*, J. Stat. Phys. **77**, 145 (1994).
8. *H.P. Fischer, Ph. Maass, W. Dieterich*, Phys. Rev. Lett. **79**, 893 (1998).
9. *H.P. Fischer, Ph. Maass, W. Dieterich*, Europhys. Lett. **62**, 49 (1998).
10. *R. Kenzler, F. Eurich, P. Maass, B. Rinn, J. Schropp, E. Bohl, W. Dieterich*, Comput. Phys. Commun. **133**, 139 (2001).
11. *S. Puri, K. Binder*, Phys. Rev. **E49**, 5359 (1994).
12. *K. Binder, S. Puri, H.L. Frisch*, Faraday Discuss. 112, 103 (1999).

13. *I.B. Krasnyuk, R.M. Taranets, M. Chugunova*, J. Adv. Res. Appl. Math. **7**, 1 (2015).
14. *I.B. Krasnyuk*, Int. J. Comput. Mater. Sci. Eng. **4**, 1550023 (2015).
15. *H. Gleiter*, Acta Mater. **48**, 1 (2000).

И.Б. Краснюк, Т.Н. Мельник, В.М. Юрченко

ПОВЕРХНОСТНО-ОРИЕНТИРОВАННЫЕ ДВУМЕРНЫЕ ВОЛНОВЫЕ СТРУКТУРЫ В ОГРАНИЧЕННЫХ БИНАРНЫХ СМЕСЯХ

Рассмотрен процесс формирования двумерных структур в бинарных смесях (например, в полимерных смесях или бинарных сплавах), который описывается уравнением Кана–Хиллиарда с динамическими граничными условиями с обратной связью. Исследованы асимптотические периодические импульсные структуры релаксационного, предтурбулентного и турбулентного типов с конечным, исчисляемым или неисчисляемым числом разрывов на периоде. Установлено, что асимптотические структуры являются элементами аттрактора в начально-краевой задаче. Рассмотрено приложение модели к случаю бинарного полимера квадратной формы. Выполнено компьютерное моделирование, описывающее формирование кристаллитов в расплаве.

Ключевые слова: уравнение Кана–Хиллиарда, нелинейные динамические граничные условия, ограниченные распределения релаксации, предтурбулентный и турбулентный тип

Рис. 1. Предельные распределения релаксационного типа в 2D-случае

Рис. 2. Предельные распределения предтурбулентного типа со счетными точками разрывов на периоде