RENORMALIZATION–GROUP ANALYSIS OF PHASE TRANSITIONS IN FRUSTRATED JOSEPHSON–JUNCTION ARRAYS

S. A. Antonenko and A. I. Sokolov

Department of Physical Electronics, Electrotechnical University, Prof. Popov Str. 5, St. Petersburg, 197376, Russia

The critical behaviour of the model which describes zero-temperature phase transitions in fully frustrated Josephson-junction arrays is studied within the field-theoretical renormalization-group approach in three dimensions. Three-loop expansions for β-functions and critical exponents are obtained. They are resummed, and specific symmetry properties of the model are employed to choose the most adequate resummation technique as well as to estimate its numerical accuracy. The fixed points of the renormalization group equations are found. Domains of continuous phase transitions are shown to exist at the phase diagram of the model, and the high precision estimates for critical exponents are obtained. These numbers, however, can hardly be considered as critical exponents for real fully frustrated arrays since corresponding initial values of coupling constants lie outside the regions of attraction of the stable fixed points. Phase transitions in such systems are shown to be, in principle, discontinuous.

The aim of the report is to study the critical behaviour of two-dimensional regular Josephson-junctions arrays (JJAs) fully frustrated with external magnetic field which undergo phase transitions from insulating to superconducting state at zero temperature.

Originally, newly developed methods of fabricating regular JJAs with well controlled parameters have given rise to recent interest to experimental and theoretical study of such systems. Main features of JJAs behaviour are known to be described by the following Hamiltonian [1–3]:

$$H = -\frac{E_c}{2}\sum_i \left(\frac{d}{d\theta_i}\right)^2 - E_j \sum_{<ij>} \cos\left(\theta_i - \theta_j - A_{ij}\right), \qquad (1)$$

where θ_i is the phase of superconducting order parameters in i-th grain (island),

$$A_{ij} = \frac{2\pi}{\Phi_0}\int_i^j \vec{A}\, d\vec{l}, \qquad (2)$$

$\vec{A}$ being the vector potential of external magnetic field, and Φ_0 is a quantum of flux. Here E_c plays a role of charging energy which is responsible for the Coulomb blockade of superconducting grains while the Josephson coupling E_j favours establishing of the global phase coherence and overall superconductivity in the system. At zero temperature superconductor-to-insulator transition occurs when the ratio of coupling constants E_c and E_j exceeds a critical value. Since quantum fluctuations are essential in the case considered, the effective dimensionality of the system should be put equal to three: $D = 2 + 1$ [2].

If external magnetic field is uniform JJAs turns out to be regulary frustrated with the frustration parameter $f = Ba_0/\Phi_0$, a_0 being the area of an elementary plaquette. We shall consider JJAs with square and traingular lattices in a magnetic field corresponding to $f = 1/2$ which are usually referred to as fully frustrated JJAs. The structure of the ground state of the model Eq. 1 with $f = 1/2$ is well known [1,2]. To study phase transitions in this model and its critical behaviour proper Hubbard-Stratonovich transformation [1,3] may be applied to Eq. 1 which results in the following Landau–Wilson Hamiltonian:

$$H = \frac{1}{2}\int d^3x \Big[m_0^2 \Phi_\alpha \Phi_\alpha^* + \nabla\Phi_\alpha \nabla\Phi_\alpha^* + (u_0/2)\Phi_\alpha \Phi_\alpha^* \Phi_\beta \Phi_\beta^* +$$

$$+ (v_0/2)\Phi_\alpha \Phi_\alpha \Phi_\alpha^* \Phi_\alpha^* + (w_0/2)\Phi_\alpha \Phi_\alpha \Phi_\beta^* \Phi_\beta^* \Big] , \tag{3}$$

where Φ_α is a complex vector field, $\alpha, \beta = 1, 2$. For $u_0 > 0$, $v_0 = 0$, $w_0 > 0$ this Hamiltonian describes phase in JJAs with square plaquettes while the case $u_0 > 0$, $v_0 < 0$, $w_0 = 0$ corresponds to JJAs with triangular lattice [1,3]. It governs also the critical behaviour of triangular JJAs with $f = 1/4$ since they are known to belong to the same universality class as fully frustrated systems with square lattice [1]. Moreover, this field-theoretical model describes phase transitions in tetragonal and hexagonal superconductors with d-wave pairing [4], in superconductors with two — s and d — order parameters [5], in some anisotropic ferro- and antiferromagnets [6,7] and in superfluid helium-3 [8].

The critical thermodynamics of the model Eq. 3 has been studed by many scientists [3,6–11]. The main attention, however, has been paid to certain limits ($v_0 = 0$, $w_0 = 0$, etc.), and the theatment has been restricted mostly with the lowest orders in ε- and $(1/N)$-expansions. On the other hand, two-loop renormalization group (RG) calculations in three dimensions show [12] that taking into account of higher-order contributions to the β-functions changes the results of the lowest-order RG analysis drastically. In particular, it alters the total number of fixed points and avoids degeneracy of the 0(4)-symmetric fixed point which is four-fold degenerate within the one-loop approximation.

In such a situation higher-order calculations seem to be very desirable since results expected would verify (or disprove) predictions of the two-loop RG analysis. Moreover, they would help to clear up to what extent the field-theoretical RG approach in three dimensions can play a role of regular (converging) approximation scheme in the case of the model with three coupling constants, provided an adequate resummation procedure is applied to the expansions employed. It should be noted that for the simple O(n)-symmetric model this technique enables one to calculate the fixed point coordinates and critical exponents with excellent accuracy [13,14]. It provides also rather good qualitative and quantitative results for models possessing two coupling constants [15–18].

We calculate the β-functions for the Hamiltonian Eq. 3 up to three-loop order within a massive theory. When expressed in terms of dimensionless effective coupling constants u, v and w are as follows:

$$\beta_u = u - u^2 - \frac{2}{3}\left(uv + uw + w^2\right) + \frac{1}{486}\Big[177u^3 + 200u^2(v + w) +$$

$$+ 46uv(v + 2w) + 308uw^2 + 144(v + w)w^2\Big] - 0.25519966u^4 -$$

$$- 0.44259575u^3(v + w) - 0.26196454u^2v(v + 2w) - 0.91607904u^2w^2 -$$

$$- 0.05845000uv^2(v + 3w) - 0.80466103uvw^2 - 0.68776103uw^3 -$$

$$- 0.12780132vw^2(v + 2w) - 0.08467496w^4 , \tag{4a}$$

$$\beta_v = v\Big[1 - u - \frac{5v + 8w}{6} + \frac{1}{486}\Big(231u^2 + 136v^2 + 236w^2 + 362uv +$$

$$+ 524uw + 380vw\Big) - 0.35690502u^3 - 0.83435956u^2v - 1.19600812u^2w -$$

$$- 0.67052441uv^2 - 1.82324690uvw - 1.25447756uw^2 - 0.18224360v^3 -$$

$$- 0.69969227v^2w - 0.88666803vw^2 - 0.34590566w^3\Big] , \qquad (4b)$$

$$\beta_w = w\Big\{1 - u - \frac{v + w}{3} + \frac{1}{486}\Big[231u^2 + 28v(v + 2w) + 20w^2 +$$

$$+ 200u(v + w)\Big] - 0.35690502u^3 - 0.47271099u^2(v + w) -$$

$$- 0.18832632uv(v + 2w) - 0.29008138uw^2 - 0.02928214v^2(v + 3w) -$$

$$- 0.14517440vw^2 - 0.08661012w^3\Big\} . \qquad (4c)$$

Such expansions for the model with arbitrary dimensionality N of the field Φ_α have been presented earlier [19].

The series Eqs. 4 are known to be at best asymptotic. To make them convergent the Borel summation technique is usually applied. The Borel transforms of the original series may be "summed up", i.e. approximated with some finite analytic expressions, in several different ways. One can construct, using a resolvent series

$$F(u, v, w, \lambda) = \sum_k \lambda^k \sum_l \sum_m a_{l,m,k-l-m} u^l v^m w^{k-l-m} , \qquad (5)$$

the generalized Pade approximants $[L/M]$, the so called Canterbury approximants invented by Chisholm, etc. (see Refs. 17 for detail). In such a situation it becomes necessary to determine which approximation scheme is the most adequate one. So, certain criteria should be formulated. The following criteria seem to be reasonable:

i) the resumation technique chosen should not lead to unphysical results;

ii) all (known) symmetries of the problem should be preserved by the approximation scheme employed;

iii) new results should be consistent with the most accurate numerical estimates for $O(n)$-symmetric and other simple models known up today;

iv) new results should be self-consistent, i.e. numerical values of any critical exponent calculated by means of the resummation of different expansions, say, expansions for γ and γ^{-1}, should be identical (as close as possible).

It may be shown that, according to these criteria, [3/1] generalized Pade-Borel approximant provides the best results. The fixed point coordinates given by this resummation procedure are presented in Table 1, which contains also, for comparison, their counterparts found earlier [12] within two-loop approximation.

Two-loop contributions to the RG functions have been mentioned to alter significantly the results of the lowest-order RG analysis. As one can see from Table 1, the taking into account of three-loop terms does not cause strong changes in the fixed point coordinates. Nevertheless, really three-loop calculations improve the results markedly.

TABLE 1. Coordinates of the fixed points of RG equations obtained within two-loop (approximant [2/1], upper lines) and three-loop ([3/1], lower lines) approximations. Some numbers presented here slightly differ from those published in Ref. 19 because of a small numerical error in early calculations.

	1	2	3	4	5	6	7	8
u_c	0 0	1.486 1.367	0 0	0.034 0.187	1.870 1.683	1.833 1.679	1.870 1.683	1.833 1.679
v_c	0 0	0 0	1.870 1.684	1.833 1.491	0 0	0 0	-1.870 -1.680	-1.359 -1.495
w_c	0 0	0 0	0 0	0 0	-0.935 -0.842	-0.680 -0.748	0.935 0.840	0.680 0.748

The point is that the model Eq. 3 possesses some specific symmetry. Indeed, if the field Φ_α undergoes the transformation

$$\Phi_1 \to \Phi_1 , \quad \Phi_2 \to i\Phi_2 , \tag{6}$$

the coupling constants are also transformed:

$$u \to u , \quad v \to v + 2w , \quad w \to -w \tag{7}$$

but the structure of the Hamiltonian itself remains unchanged [12]. Just the same situation takes place in the case of another field transformation [3,7]:

$$\Phi_1 \to 2^{-1/2}(\Phi_1 + i\Phi_2) , \quad \Phi_2 \to 2^{-1/2}(i\Phi_1 + \Phi_2) \tag{8}$$

which does not affect the Hamiltonian structure resulting only in the following replacement of u, v and w:

$$u \to u + v + 2w , \quad v \to -2w , \quad w \to -v/2 . \tag{9}$$

It is well known that RG functions of the problem are completely determined by a structure of the Hamiltonian; they do not depend on u_0 , v_0 and w_0 which play a role of initial values of effective coupling constants when the RG flow of u, v and w is searched. Hence, RG equations should be invariant with respect to any transformation conserving a structure of the Hamiltonian [20]; Eqs. 7,9, in particular, were shown to be such transformations.

It means that β_u, β_v , and β_w should obey some special symmetry relations which may be readily written down:

$$\begin{aligned} \beta_u(u,v,w) &= \beta_u(u,v + 2w,-w) , \\ \beta_v(u,v,w) + 2\beta_w(u,v,w) &= \beta_v(u,v + 2w,-w) , \\ \beta_w(u,v,w) &= -\beta_w(u,v + 2w,-w) . \end{aligned} \tag{10}$$

$$\begin{aligned} \beta_u(u,v,w) + \beta_v(u,v,w) + 2\beta_w(u,v,w) &= \beta_u(u + v + 2w,-2w,-v/2), \\ \beta_v(u,v,w) &= -2\beta_w(u + v + 2w,-2w,-v/2) , \\ 2\beta_w(u,v,w) &= - \beta_v(u + v + 2w,-2w,-v/2). \end{aligned} \tag{11}$$

Moreover, due to this special symmetry, transformations Eqs. 7, 9 can, at most, rearrange the fixed points of RG equations not affecting numbers u_c, v_c and w_c themselves. This is precisely what occurs when one applies Eq. 7 to the content of Table 1: the first four fixed points stay at their places while the points 5–8 undergo pair transpositions $5 \to 7$, $7 \to 5$, $6 \to 8$, $8 \to 6$. So, with the symmetry mentioned in mind, the 5-th and 7-th (the 6-th and the 8-th) points may be thought to be actually the same fixed point. Due to this symmetry any statement concerning the behaviour of the system which is correct for $v = 0$ should be correct as well for $2w = -v$.

What happens when another transformation is applied to fixed point coordinates presented in Table 1? It is easy to see that Eq. 9 practically does not change the location of fixed points 1, 2, 7 and 8 and causes pair transposition $3 \to 5$, $5 \to 3$. The rest of fixed points, the 4-th and the 6-th ones, however, are converted one to another under Eq. 9 only within three-loop approximation. Corresponding two-loop results turn out to violate the symmetry relations induced by Eq. 9. More precisely, the differences between the coordinates of the point 4 and the transformed coordinates of the point 6 ("symmetry discrepancies") given by [2/1] Pade–Borel approximants are about 0.3, while within three-loop approximation they are substantially less than 0.01.

Comparison of coordinates of 0(2)- and 0(4)-symmetric fixed points given by [3/1] approximants with "exact" ones [13,14] shows that they differ by no more than 0.01. Since symmetry discrepancies obtained within this approximation scheme are of the same order of magnitude, we believe that the procedure mentioned provides an accuracy of order of one percent for models with several coupling constants. To the contrary, two-loop RG expansions resummed in the same way lead in such cases, as we have seen, to rather crude estimates. So, the calculation of three-loop terms enables one to obtain results which are much more accurate than those given by two-loop RG expansions.

All the fixed points found are unstable in three-dimensional parameters space (u,v, w). The 4-th and the 6-th ones, however, are stable wtithin the planes (u, v) and (u, w) respectively. The existence of such points is important since it implies the possibility of continuous phase transitions in numerous physical systems described by the model Eq. 3. Corresponding critical exponents may be found by calculation of three-loop RG expansions, say, for γ and η with subsequent resummation of γ series and making use of well-known scaling relations. Three-loop RG series for γ^{-1} and η are as follows:

$$\eta = \frac{1}{243}\left(3u^2 + 2v^2 + 4w^2 + 4uv + 4uw + 4vw\right) + \\ + 0.00011428\left[9u^3 + 18u^2(v + w) + 15uv(v + 2w) + \right. \\ \left. + 24uw^2 + 5v^2(v + 3w) + 18vw^2 + 8w^3\right], \qquad (12)$$

$$\gamma^{-1} = 1 - \frac{u}{4} - \frac{v + w}{6} + \frac{1}{72}\left(3u^2 + 2v^2 + 4w^2 + 4uv + 4uw + 4vw\right) - \\ - 0.00228954\left[9u^3 + 18u^2(v + w) + 15uv(v + 2w) + 24uw^2 + \right. \\ \left. + 5v^2(v + 3w) + 18vw^2 + 8w^3\right] - 0.00089125\left[9u^3 + 18u^2(v + w) + \right. \\ \left. + 14uv(v + 2w) + 20uw^2 + 4v^2(v + 3w) + 16vw^2 + 8w^3\right]. \qquad (13)$$

Analogous expressions for arbitrary field dimensionality have been presented in Ref. 19.

Critical exponents for the fixed point 4 stable within the plane (u, v) are found to be:

$$\gamma = 1.336\,, \quad \eta = 0.026\,, \quad \nu = 0.677\,, \quad \alpha = -0.030\,, \quad \beta = 0.347\,. \qquad (14)$$

The calculation of critical exponents for the fixed point 6 stable within the plane (u, w) gives values which are identical, within the accurracy adopted, to those of the point 4. This is the manifestation of the specific symmetry discussed in detail earlier.

It is interesting to compare the results obtained for $O(2)$-symmetric fixed point (the point 3, unstable) with data given by resummed six-loop RG expansions for $O(n)$-symmetric $\lambda\Phi^4$ model [13,14]. The six-loop approximation being nowadays the most advanced instrument for calculation of critical exponents in three dimensions leads, for $n = 2$, to

$$\gamma = 1.316\,, \quad \eta = 0.032 \qquad (15)$$

while the approximation scheme employed gives:

$$\gamma = 1.310\,, \quad \eta = 0.026\,. \qquad (16)$$

The numbers Eq. 16 are also practically identical to those obtained for the fixed point 5. It should be thought as another manifestation of the symmetry just mentioned.

The critical exponent ν is known to control the superfluid density ρ_s, the Mott gap and the crossover temperature to classical behaviour in JJAs [3]. That's why its numerical value is of prime importance in our case. As follows from above calculations it should be equal to 0.68, i.e. to the value which is very close to that for unfrustrated arrays.

On the other hand, the numbers Eq. 14 can hardly be considered as critical exponents for real fully frustrated JJAs since corresponding initial values u_0, v_0, w_0 of coupling constants lie outside the domains of attraction of the stable fixed points. More precisely, these values are situated in the region of fluctuation instability of the model Eq. 3. Hence, superconductor-to-insulator phase transition in fully frustrated JJAs should be, in principle, discontinuous (first-order).

It is well known, however, that fluctuation-induced first-order phase transitions are very weak. Therefore, before a discontinuous transition will occur, a system usually demonstrates scaling-like behaviour governed by effective critical exponents. Since in the case considered the $O(4)$-symmetric fixed point lies on the boundary of the domains in (u, v) and (u, w) planes which contain proper initial values of coupling constants, fully frustrated JJAs are expected to behave, in the critical region, in a way similar to that of $O(4)$-symmetric model. Corresponding critical exponent values calculated on the base of higher-order RG expansions in three dimensions are as follows [21]:

$$\gamma = 1.441\,, \quad \eta = 0.032\,, \quad \nu = 0.732\,, \quad \alpha = -0.197\,, \quad \beta = 0.378\,. \qquad (17)$$

These numbers differ substantially from those for JJAs in zero magnetic field.

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