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CRITICAL COMMENTS ON THE THEORETICAL INTERPRETATION OF VAVILOV-CHERENKOV EFFECT

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We investigate the theory of the Vavilov-Cherenkov effect on the basis of the new, found by us, solution of system of the Maxwell's equations with a uniform rectilinear moving source. We make the comparison with the Sommerfeld, Tamm and Frank's theory at each stage of our calculations. The applicability of the condition of radiation $\cos\vartheta = c/nv$ *is discussed. We show that this condition is a necessary condition, but not sufficient for the occurrence of radiation.*

1. Introduction

The theory of the Vavilov-Cherenkov (VC) effect is closely connected to problem of a finding the fields produced by a pointlike uniformely moving charge. To formulate this problem, it suffices to solve Maxwell's equations (ME). Since Maxwell has formulated the famous equations (1861) and Hertz and Heaviside have given to them a modern kind (1893), they at once have attracted attention of outstanding physicists of the XIX and XX centuries. Especially it is necessary to note the Sommerfeld's paper [1] devoted to the problem of VC effect in vacuum. This work after almost 30-year's silence was quoted by Tamm [2] and both Tamm and Frank (TF) [3]. It is considered that the theory of the field generated by an uniformly moving source is a key problem in electrodynamics. Rather full bibliography on the theoretical and experimental discussion of VC effect can be found in [4]. It would be desirable to note such fact: among set of papers devoted to VC effect there are few critical. In this connection, we shall specify Zin's paper [5] in which electrical and magnetic fields obtained in the works [1-3] are analyzed. The importance of this paper has been noted by Zrelov in [6].

The purpose of our work is to give an alternative method of the consideration of VC effect and to show that it leads to a deeper understanding of the problem as a whole.

2. Review of Maxwell's equations and their solutions

We shall write down ME for determination of the strengths generated by a pointlike charge which moves in the isotropic homogeneous medium with the constant refraction index n

$$\mathrm{rot}\,\mathbf{E} + (1/c)(\partial/\partial t)\mathbf{H} = 0, \qquad \mathrm{div}\,\mathbf{H} = 0, \qquad \mathrm{rot}\,\mathbf{H} - (n^2/c)(\partial/\partial t)\mathbf{E} = (4\pi/c)\rho\mathbf{v}, \qquad \mathrm{div}\,\mathbf{E} = 4\pi\rho/n^2 . \tag{1}$$

If the refraction index n is constant, then an idealized case is assumed. It does not correspond to a real dispersive medium in which the refraction index n is the operator and in momentum representation depends on frequency of an electromagnetic field. It is possible to disagree with this point of view, because if there is no solution especially in that specific case, there is no solution in a more general case. More shortly, if there is a solution of an equation in the specific case, it is a necessary condition of existence of the solution general.

In ME (1) the charge density ρ and current density $\rho\mathbf{v}$ are given. We consider the case when the charge density has the form

$$\rho = e\delta(x_1)\delta(x_2)\delta(z - vt), \quad \mathbf{v} = (0,0,v). \tag{2}$$

This density corresponds to a pointlike charge moving along given z-direction.

Let's write out potentials (solution) of the ME (1) appropriate to this density and current, which have been found in Sommerfeld, Tamm and Frank(STF) papers [1-3]

$$\varphi = (2e/n^2)\theta(vt - z - \gamma\sqrt{x_1^2 + x_2^2})/R, \, A_z = n\varphi v/u, \quad u = c/n \tag{3}$$

where the notations are introduced: $R = \sqrt{(vt - z)^2 - \gamma^2(x_1^2 + x_2^2)}$, θ is the Heaviside step function and $\gamma = \sqrt{v^2/u^2 - 1}$.

Even not performing calculations, we can be sure that the potentials of Eq.(3) don't satisfy the wave equation (D'Allembert's equation) with the source (2). It occurs because in ME (1) the density of a charge and the density of a current are determined at any velocity of a source, and the potentials of Eq.(3) are determined at velocity v of a source more than velocity of light in medium ($v > c/n$ or $v > c$ in vacuum). Nevertheless, with the help of such unsatisfactory potentials the strengths, which are somewhat criticized by Zin [5], are calculated

$$\mathbf{E} = -\operatorname{grad}\varphi - (1/c)\partial\mathbf{A}/\partial tt, \quad \mathbf{H} = \operatorname{rot}\mathbf{A}$$

or taking into account the properties of the potentials of Eq.(3) we have

$$E_i = -\partial\varphi/\partial x_i, \quad i = 1,2; \quad E_z = \gamma^2\partial\varphi/\partial z, \quad \mathbf{H} = (1/c)(\mathbf{v}\times\mathbf{E}).$$

It is worth noting that in Zin's paper [5], in contrast to STF papers [1-3], the strengths consist of two parts. One part contains the Heaviside step function, and another one contains the derivative with respect to this step, that is delta function. These parts are denoted by $\mathbf{E}^{(1)}$ and $\mathbf{H}^{(1)}$ and $\mathbf{E}^{(0)}$,$\mathbf{H}^{(0)}$

$$\mathbf{E} = \mathbf{E}^{(1)} + \mathbf{E}^{(0)}, \quad \mathbf{H} = \mathbf{H}^{(1)} + \mathbf{H}^{(0)}$$

where $\mathbf{E}^{(1)}$ and $\mathbf{H}^{(1)}$ are called in [5] as pulsed electric and magnetic fields. They appear due to the presence of step function θ in the potentials of Eq.(3) and they are much more important than the fields $\mathbf{E}^{(0)}$ and $\mathbf{H}^{(0)}$. The latter ones coincide with STF-fields [1-3]. If we introduce the cylindrical coordinates with the basic unit vectors

$$\mathbf{x}^0 = (x_1/x_\perp, x_2/x_\perp, 0), \quad \psi^{0)} = (-x_2/x_\perp, x_1/x_\perp, 0), \quad \mathbf{z}^0 = (0,0,1),$$

where $x_\perp = \sqrt{x_1^2 + x_2^2}$, then the pulsed fields $\mathbf{E}^{(1)}$ and $\mathbf{H}^{(1)}$ [5] are

$$\mathbf{E}^{(1)} = (2e/n^2)\gamma\left(\mathbf{x}^{(0)} - \gamma\mathbf{z}^{(0)}\right)\delta(vt - z - \gamma x_\perp)/R; \quad \mathbf{H}^{(1)} = (nv/u)\mathbf{z}^{(0)} \times \mathbf{E}^{(1)}. \tag{4}$$

As it has been stated in [5], precisely, the fields $\mathbf{E}^{(1)}$ and $\mathbf{H}^{(1)}$ have to be responsible for radiation. It is rather easy to verify that electric strengths in the STF theory[1-3] as well as in [5] don't satisfy the Gauss' theorem

$$\int \operatorname{div} \mathbf{E} \, d^3 r = \oint \mathbf{E}^{(1)} \, d\boldsymbol{\sigma} \neq 4\pi e / n^2 ,$$

where $d\boldsymbol{\sigma} = \mathbf{r}^{(0)} r^2 \sin\vartheta d\vartheta d\psi$ is the surface element and

$$\mathbf{r}^{(0)} = \mathbf{x}^{(0)} \sin\vartheta + \mathbf{z}^{(0)} \cos\vartheta ,$$

where ϑ is the polar angle between the directions of z- axis and radius $\mathbf{r}$.

Therefore we come to the conclusion that at velocity of a source v greater than velocity of light in medium c/n (or $v \geq c$ in vacuum) ME (1) don't have solutions [7]. The latter does not mean that the sources moving with velocity greater than velocity of light in medium (or in vacuum) are forbidden, and is a reflection of the fact that if a charge(source) goes with velocity exceeding velocity of light, it does not create its own magnetic and electric field.

We have found in [7] a new solution of ME (1) when velocity of a source $v = c/n$ (or $v = c$ in vacuum). The fields corresponding to such solution can be written (similarly to [5])

$$\mathbf{E} = \mathbf{E}^{(\mathrm{rad})} + \mathbf{E}^{(\mathrm{st})}, \qquad \mathbf{H} = \mathbf{H}^{(\mathrm{rad})} + \mathbf{H}^{(\mathrm{st})}$$

where the electric field $\mathbf{E}^{(\mathrm{rad})}$ and the magnetic field $\mathbf{H}^{(\mathrm{rad})}$ are transverse with respect to radius vector $\mathbf{r}$: $\mathbf{r}\mathbf{E}^{(\mathrm{rad})} = \mathbf{r}\mathbf{H}^{(\mathrm{rad})} = 0$, and they are called by us as radiative fields. They take the form

$$\mathbf{E}^{(\mathrm{rad})} = \mathbf{a} e[\delta(t - r/u)/n^2 ur]\theta(1 - \cos\vartheta) ;$$
$$\mathbf{H}^{(\mathrm{rad})} = \boldsymbol{\psi}^{(0)} (ut/x_\perp) e[\delta(t - r/u)/n^2 ur]\theta(1 - \cos\vartheta) , \tag{5}$$

where vector $\mathbf{a} = \mathbf{x}^{(0)} z/x_\perp - \mathbf{z}^{(0)} = \mathbf{x}^{(0)} \cot\vartheta - \mathbf{z}^{(0)} = 0$ is orthogonal to the radius vector $\mathbf{r}$: $\mathbf{a}\mathbf{r} = 0$.

It is easy to verify that [7]

$$\mathbf{H}^{(\mathrm{rad})} = n(\mathbf{r}^0 \times \mathbf{E}^{(\mathrm{rad})}), \qquad |\mathbf{E}^{(\mathrm{rad})}| = (1/n)|\mathbf{H}^{(\mathrm{rad})}|.$$

The static fields $\mathbf{E}^{(\mathrm{st})}$ and $\mathbf{H}^{(\mathrm{st})}$ are equal to

$$\mathbf{E}^{(\mathrm{st})} = \mathbf{x}^{(0)} e[\delta(ut - z)]/n^2 x_\perp = \mathbf{x}^{(0)} e[\delta(ut/r - \cos\vartheta)]/r^2 \sin\vartheta ;$$
$$\mathbf{H}^{(\mathrm{st})} = \boldsymbol{\psi}^0 (\mathbf{E}^{(\mathrm{st})} \mathbf{x}^0) n . \tag{6}$$

At large distances they decrease similarly to Coulomb potentials. It is easy to verify that our new field $\mathbf{E}$ satisfy Gauss' theorem [7]

$$\oint \mathbf{E}^{(\mathrm{st})} \, d\boldsymbol{\sigma} = (2\pi e / n^2) \int_0^\pi \delta(ut/r - \cos\vartheta) \sin\vartheta \, d\vartheta = 4\pi e / n^2$$

It is interesting to consider the behavior of these found fields of Eqs. (5) and (6) under Lorentz transformations (LT). To this end it is convenient to write the electric and magnetic strengths in Cartesian coordinate system. For simplicity we consider the case when $n = 1$ (vacuum). We replace $u \to c$ in Eqs. (5) and (6). Under the LT of the coordinates

$$x_1 = x_1', \quad x_2 = x_2', \quad z = (z' + vt')/\sqrt{1 - v^2/c^2}, \quad ct = (ct' + (v/c)z')/\sqrt{1 - v^2/c^2},$$

where v is the velocity of an inertial reference frame (IRF). The electric field E_z and magnetic field H_z are invariant, that is

$$E_z = E_{z'} = -e[\delta(t' - r'/c)/cr']\theta(1 - \cos\vartheta'); \quad H_z = H_{z'} = 0,$$

where

$$r' = \sqrt{x_1'^2 + x_2'^2 + z'^2}.$$

The fields E_i and H_i (i=1,2) are respectively transformed as

$$E_1 = (E_1' + (v/c)H_2')/\sqrt{1 - v^2/c^2}, \quad E_2 = (E_2' - (v/c)H_1')/\sqrt{1 - v^2/c^2};$$

$$H_1 = (H_1' - (v/c)E_2')/\sqrt{1 - v^2/c^2}, \quad H_2 = (H_2' + (v/c)E_1')/\sqrt{1 - v^2/c^2}.$$

It is well known that the formulas of inverse transformations are similar in form with **E** and **H** replaced by **E′** and **H′** and v by $-v$. The fields **E** and **H** (see Eqs.(5) and (6)) (or **E′** and **H′**) don't contain velocity v of an IRF and therefore it is impossible to eliminate these fields by any transformation of coordinates, in contrast to the Lienard- Wiechert fields generated by a uniformly moving source which depend on the choice of IRF. Our new fields [7] exist in any IRF.

3. Energy and momentum of VC radiation

From the ME (1) follows the energy balance (law of the conservation of energy) which in integral form is represented as

$$\int d^3 r(\partial/\partial t)\left(n^2E^2 + H^2\right)/8\pi + \int d^3 r\rho\mathbf{vE} = -(c/4\pi)\oint d\,\boldsymbol{\sigma}, \mathbf{E}\times\mathbf{H}. \tag{7}$$

We shall start analysis of this equation with the term which includes the Poynting vector flux

$$-(c/4\pi)\oint d\,\boldsymbol{\sigma}, \mathbf{E}\times\mathbf{H} = -(c/4\pi)\oint d\,\boldsymbol{\sigma}, \mathbf{E}^{(\mathrm{rad})}\times\mathbf{H}^{(\mathrm{rad})} + \ldots \tag{8}$$

where dots denote that we neglect the terms which fall off with the distance more rapidly than $1/r^2$. Taking into account Eqs. (5) and (6) we obtain

$$(c/4\pi)\left(\mathbf{E}^{(\mathrm{rad})}\times\mathbf{H}^{(\mathbf{rad})}\right) = (cn/4\pi)\mathbf{r}^0\mathbf{E}^{(\mathrm{rad})2} > 0.$$

Denoting the expression of Eq.(8) as the change of the lost energy (after integration over angle variables) we have

$$-(\partial/\partial t)W^{(\mathrm{lost})} = (e^2/n^2u)\left(\delta(t - r/u)\right)^2 A \tag{9}$$

where A is an infinite positive and is equal to

$$A = \int_0^\pi \mathrm{d}\vartheta / \sin\vartheta = -\ln \tan^2 \varepsilon / 2 \qquad \text{at } \varepsilon \to +0,$$

and the square of the δ- function is understood in terms of cutting parameter ω which has the dimensionality of the frequency. The positive constant A can be included in the cutting parameter ω. We can calculate the lost energy per time $t = d/u$ where d is the way covered by the charge. Integrating Eq.(9) we obtain

$$W^{(\text{lost})} = (e^2\omega^2 d)/c^2. \qquad (10)$$

We can also find the lost momentum **G** using the law of conservation of momentum (Newton's second law)

$$(\partial/\partial t)\left(\mathbf{G}^{(\text{source})} + \mathbf{G}^{(\text{field})}\right) = \int \mathrm{d}^3 r \rho \left(\mathbf{E} + (1/c)\mathbf{v}\times\mathbf{H}\right). \qquad (11)$$

Sinse the source of Eq.(2) is uniformly moving therefore $(\mathrm{d}/\mathrm{d}t)\mathbf{G}^{(\text{source})} = 0$ and Eq.(11) takes the form

$$(\partial/\partial t)\mathbf{G}^{(\text{field})} = \mathbf{z}^0 \int \mathrm{d}^3 r \rho E_z^{(\text{rad})} < 0. \qquad (12)$$

We have shown in [7] that the static fields $\mathbf{E}^{(\text{st})}$ and $\mathbf{H}^{(\text{st})}$ give no the contribution to the Lorentz force (LF). From Eq.(12) it follows that the LF is the breaking force and it has direction along z- axis and is antiparallel to the charge's velocity. The amount of lost momentum per time t=d/u is determined as

$$G_z^{(\text{field})} = (e^2 k^2/n^2)(d/u) \qquad (13)$$

where k is the cutting parameter too.

If we assume that between energy of Eq.(10) and momentum of Eq.(13) the relation

$$G = G_z^{(\text{field})} = W^{(\text{lost})}/u$$

is valid then it is necessary and sufficiently for the cutting parameters ω and k to obey the relation

$$k^2/n^2 = \omega^2/c^2 \quad \text{or} \quad \omega = kc/n = ku. \qquad (14)$$

Relation (14) is valid for electromagnetic waves in medium only. Therefore we may call the lost energy and field momentum as the energy and momentum of radiation, that is

$$W^{(\text{lost})} = W^{(\text{rad})}; \quad G^{(\text{field})} = G^{(\text{rad})}.$$

In STF theory [1-3] the lost energy and field momentum are equal respectively (in terms of cutting parameters ω and k)

$$W^{(\text{lost})} = (e^2\omega^2 d)/c^2(1 - u^2/v^2); \quad G_z^{(\text{field})} = (e^2 k_z^2 dv/c^2)(1 - u^2/v^2).$$

Repeating reasonings similar to preceding ones we find the relation which connects the cutting parameters ω and k_z

$$G_z^{(\text{field})} = W^{(\text{lost})}/v \Rightarrow \omega^2 = k_z^2 v^2 \quad \text{or} \quad \omega = k_z v = kv\cos\vartheta \neq kc/n.$$

This dispersion relation does not describe the propagation of electromagnetic waves. This means that the condition of radiation in STF theory[1-3], namely, $v\cos\vartheta = c/n$

cannot be interpreted as radiation. It can be interpreted as a necessary condition of the radiation only.

4. Conclusion

From what has been said above the following conclusion may be drawn: the uniformly moving charge radiates along the direction of its own movement in the only case, when its velocity v comes nearer to spherical velocity of the propagation of electromagnetic waves ($v \to c-0$ or $v \to u-0$ in medium). The medium is not determinative in a theoretical explanation of VC effect. This effect has resonant character, instead of cooperative. VC effect can be observable in vacuum and also in medium with the refraction index smaller than 1.

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