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INVARIANTS

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*The paper consists of two parts: (1) "**As a foreword: A tiny miscellany of reminiscences**" - the dedication to K.B. Tolpygo and V.A. Telezhkin; and then (2) **Algebraic Invariants And Quantum Mechanics,** where a program for constructing the nonclassical algebraic structures and a quantization scheme are proposed by analogy with the Klein geometric program. In this context the group of affine canonical transformations is considered in detail. An application to theory of nonlinear wave equations represented as a pair of 'p'-polynomials on a nonclassical algebraic structure is outlined.*

As a foreword: A tiny miscellany of reminiscences

The destiny happily has disposed in such a manner that K.B. Tolpygo (KB) was the first actively and successfully working professional physicist who to me could be seen. During the first meeting with the professors of the university, even before departure again of baked students of the faculty of physics to agricultural work and up to an actual beginning of educational process, KB has acted with introductory speech in which he has made an emphasis on practical value of physical education - its universality and flexibility allow the graduates with diploma in physics successfully to work both in various branches of science from mathematics to biology and in a national economy, now he would say, in business and management. The life completely has confirmed correctness of the affirmation. Now, after particular social vicissitudes, the graduates of physical faculty of the university can be seen everywhere (hardly pertinently to enumerate where exactly, everyone can make it and in such a manner that it will reflect his/her social position). Further, some years ago when I became the post-graduate student of Prof. A.A. Borghardt of the department of theoretical physics, I has experienced this universality and flexibility in action. In the department of theoretical physics everybody were rather nonpredatory symbiosed that generally was possible at that time. And in all that KB took part with lively interest and activity. It were both proper theoretical physics, and solid state physics, and biology, and it is a lot of what even to term definitely with difficulty, and plus to all the pointed, numeric and visualization methods in various applications. In connection with the last branch and not only with, Dr. V.A. Telezhkin, who tragically left us last year, deserves especially kind remembrance. He was assignee of KB at the head of the department of theoretical physics. In contrast to KB, who for me was the professor of general physics at the university, with Dr. Telezhkin it was fortunately enough to prepare one scientific work (unpublished) on ultrasonic clearing of semiconductors, perhaps, first technological application of the stochastic resonance phenomena. I think this work is more actual now then four years back, but, unfortunately, at present it is only my trouble to prepare a final draft.

In remembrance of the beautiful days, - The paper below has been discussed at the time of the well-known KB's seminars.

Algebraic Invariants And Quantum Mechanics

A program of constructing nonclassical algebraic structures and quantization by analogy with the Klein geometric program is proposed. In this context the group of affine canonical transformations is considered in detail. An application to nonlinear wave equations is outlined.

1. Introduction

The development of the old quantum theory was carried out in the direction: The realization of observables of the classical mechanics and their algebra remained without modification and only realization of states was changed. As a result, the theory has been presented a wealthy of material [1]. However, the (new) quantum theory was setting off by the other way: It was constructed in such a way that the theory remained the algebraic structure of classical mechanics but fully renounced an idea of phase space and of observables as smooth functions on phase space [2,3]. Nevertheless, the idea of phase space has appeared in quantum mechanics already at a relatively early stage of its development [4,5] and then has taken clear form in the work [6], then in the theory of deformational quantization [7,8] and in a number of other works [9-12] (see [12] for further references). The Weyl-Wigner-Moyal (WWM) representation has attracted the main attention. Besides the WWM representation there are other representations of quantum mechanics in phase space including rare representations [13-16]. However, the WWM representation occupies a special position among them: In its basis lies a maximal group (more precise definition will be given in what follows). For this reason the WWM representation is the most appropriate initial point for algebro-geometric speculations.

The main Klein's idea [17] lies in the correspondence to any geometry of a group which acts in its space. In general, every group of transformations determines its own geometry. This geometry studies properties of figures which are invariant under the action of a given group transformations. By Klein, a group is the first notion of geometry and it can be interpreted as group of symmetry for geometry which is arising from. All that is well known [17,18]. The Euclidean geometry is a typical example for the group which has the structure of semidirect product of a group of orthogonal matrices and an additive group of vectors (in an n-dimensional space).

In this work we propose an application of the basic idea of geometric Klein program, which is understood in a wide sense, to the problem of constructing nonclassical algebraic structures on the set of classical observables. The starting points are the WWM representation of quantum mechanics in phase space and the group of affine canonical transformations of this phase space.

This work has by the first purpose a statement of the program in the so simple terms as far as it is possible. At least it means that we shall be limited to systems with one degree of freedom. Certainly it is not an exhausting statement. It is an occasion for discussion.

2. Classical and quantum mechanics in phase space

This section is devoted to a brief reminder of the concept of phase space in contexts of the classical and quantum mechanics [1,4-6,10,19].

2.1. Classical Mechanics

Let $x=(q,p)\in M=\mathbb{R}^2$ be a point of the phase space *M* and (*q,p*) are identified with canonical variables of a classical mechanical system, and let $\mathcal{A}$ be a set of smooth functions on *M*. The observables of a classical system are identified with elements $f,g,\dots\in\mathcal{A}$ and $\mathcal{A}$ is equipped with two algebraic structures: the pointwise multiplication (the Jordan product),

$$(f\cdot g)(x)=f(x)g(x),$$

and the Poisson bracket operation,

$$\{f,g\}=\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}=\omega^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}, \tag{2.1}$$

which makes $\mathcal{A}$ into a Lie algebra. Here and in the following the notation ω is used for the symplectic matrix.

The states of a system of the classical mechanics are probability distributions on *M*.

2.2. Phase Space Representations of Quantum Mechanics

As the basic example we shall remind the WWM representation of the quantum mechanics in phase space [5,6,21]

Let *M* and $\mathcal{A}$ be the same as in Section 2.1. For each integer $k\geq 0$ define bilinear partial Moyal bracket of the *k*-th degree

$$\{f,g\}^{(k)}=\omega^{i_1j_1}\omega^{i_2j_2}\cdots\omega^{i_kj_k}\frac{\partial^k f}{\partial x^{i_1}\cdots\partial x^{i_k}}\frac{\partial^k g}{\partial x^{j_1}\cdots\partial x^{j_k}}. \tag{2.2}$$

For $k=0$ it is usual pointwise multiplication and for $k=1$ it is the Poisson bracket (2.1).

It is useful to note that the $\{f,g\}^{(k)}$ has a form similar to the *k*-th transvection operator of the invariant theory [20,21].

The basic algebraic structures of the WWM phase space representation of the quantum mechanics, the Jordan-Moyal product and the Poisson-Moyal bracket, are of the form:

$$f\circ g=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\hbar}{2}\right)^{2n}\{f,g\}^{(2n)}, \tag{2.3}$$

and

$$\{f,g\}_{\mathrm{M}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\hbar}{2}\right)^{2n}\{f,g\}^{(2n+1)} \tag{2.4}$$

Here $(\cdot\circ\cdot)$ is a genuine structure of the Jordan algebra and $\{\cdot,\cdot\}_{\mathrm{M}}$ is a genuine structure of the Lie algebra on $\mathcal{A}$, and $\hbar$ is the Planck constant.

This representation differs from the usual operator formulation of the quantum mechanics [2,3] in form (and only in form) but is closest under the form to the classical mechanics. The algebraic structures of the classical mechanics are a limiting case of the algebraic structures (2.3) and (2.4): the pointwise product and the Poisson bracket may be derived from the Jordan-Moyal product and the Poisson-Moyal bracket by passing to the limit $\hbar \to 0$. The case of quantum states is not so simple with respect to limit $\hbar \to 0$ but it is the case of particular importance which lies outside the paper.

The algebraic structures of standard operator formulation of the quantum mechanics and the WWM representation are connected by the Weyl-Wigner correspondence rules [4]:

$$-\frac{i}{\hbar}[\mathbf{f},\mathbf{g}] \to \{f,g\}_{\mathrm{M}}; \qquad \frac{1}{2}[\mathbf{f},\mathbf{g}]_{+} \to f \circ g .$$

Here $[\mathbf{f},\mathbf{g}] = \mathbf{fg} - \mathbf{gf}$ and $[\mathbf{f},\mathbf{g}]_{+} = (\mathbf{f}+\mathbf{g})^2 - \mathbf{f}^2 - \mathbf{g}^2$.

There is a lot of correspondence rules and for every correspondence rule there is the associate phase space representation. Among them there are rare phase space representations [13-16]. For example, the representation which relates to so called Blokhintsev bracket,

$$\{f,g\}_{\mathrm{B}} = \sum_{n=1}^{\infty} \frac{(-i\hbar)^{n-1}}{n!} \omega^{ij} \frac{\partial^n f}{\partial (x^i)^n} \frac{\partial^n g}{\partial (x^j)^n} \tag{2.5}$$

It is necessary once again to note that in reminders of this section nothing was told about states of the mentioned mechanics as it makes a subject of other program [22] and our basic object of consideration are the algebraic structures. However, in the context of what follows, for example, it is easy to find the relation between the basic group and coherent states notions.

3. The problem

This section is organized as follows. Having begun with a brief reminder of the Dirac problem then, as a preparation to the following, we determine the invariance group $\mathcal{G}$ (a group of phase space transformations) of the algebraic structures of the WWM phase space representation. Then we pose a question on the solution of the Dirac problem by means of introducing into consideration of new algebraic structures on $\mathcal{A}$ instead of the algebraic structures of the classical mechanics. More precisely, these new algebraic structures are subject to definition by means of the required property due to invariance under the action of the basic group $\mathcal{G}$ of the phase space transformations. Then these new algebraic structures on $\mathcal{A}$ are subject to use for solution of the Dirac problem by means of the Weyl-Wigner correspondence.

In such way, after obvious generalization, the basic concept of quantization is the group of affine phase space transformations. The quantization amounts to the construction of relevant nonclassical invariant algebraic structures on the set of classical mechanical observables $\mathcal{A}$. It is a clear analog of the Klein geometric program [17,18].

3.1. The Dirac problem

The Dirac problem (quantization) can be formulated as a correspondence problem in the following manner [22,23]: To establish *ab initio* a mapping Q which is defined on the algebra of classical observables $f, g, \dots \in \mathcal{A}$, takes values in the algebra of quantum observables $\mathbf{f}, \mathbf{g}, \cdots$; (self-adjoint operators acting in a Hilbert space $\mathcal{H}$ [2,3]), and has the following properties:

$$\textbf{(Q1)} \quad Q(\lambda f + \mu g) = \lambda Q(f) + \mu Q(g), \quad \lambda, \mu \in \mathbb{R};$$

$$\textbf{(Q2)} \quad Q(\{f, g\}) = \frac{1}{i\hbar}[Q(f), Q(g)];$$

$$\textbf{(Q3)} \quad Q(f^2) = (Q(f))^2;$$

$$\textbf{(Q4)} \quad Q(1) = \mathbf{1}_{\mathcal{H}}.$$

It is known that such correspondence problem leads to a lot of difficulties [22,23].

3.2. The basic group of the WWM representation

We take as a starting point the set of partial Moyal brackets (2.2). The Jordan-Moyal product (2.3) and the Poisson-Moyal bracket (2.4) are the linear combinations of the partial Moyal brackets. It has been outlined above (Section 2.2.) that the partial Moyal brackets are in a way connected with transvections of the classical invariant theory. Hence, we may ask the question: If the set of partial Moyal brackets is a set of bilinear invariant algebraic structures on the set of classical observables, what should be the appropriate group of the phase space transformations? The answer to this question is almost obvious: It is group of affine canonical transformations (the general form of an affine canonical transformation is: $x \to \mathbf{C}x + \xi$, where $\mathbf{C}$ is a linear canonical transformation, and $\xi \in M$; as a group it has the structure of semidirect product of the symplectic group and the group of vectors M: $\mathcal{G}_{\mathrm{C}} = \mathrm{Sp}(M) \otimes M$. We can find this answer in the classical invariant theory (see [20,21]).

3.3. The program

A program for the solution of the Dirac problem can be formulated now in the following manner.

For a given group $\mathcal{G}$ of affine transformations of the phase space M and for a given system of observables $\mathcal{A}$, it is necessary first of all to construct a set of bilinear maps $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ invariant under the transformations of $\mathcal{G}$. Then, from these maps it is necessary to construct bilinear multiplication operations of Lie $\{\cdot,\cdot\}_{\mathcal{G}}$ and Jordan $(\cdot \circ \cdot)_{\mathcal{G}}$. If it is possible to construct these operations in a sense uniquely, then we can use them instead of classical operations of the Poisson bracket and the pointwise multiplication in the Dirac problem:

$$\textbf{(q1)} \quad q(\lambda f + \mu g) = \lambda q(f) + \mu q(g), \quad \lambda, \mu \in \mathbb{R};$$

$$\textbf{(q2)} \quad q(\{f, g\}) = \frac{1}{i\hbar}[q(f), q(g)];$$

$$\textbf{(q3)} \quad q((f \circ f)_{\mathcal{G}}) = (q(f))^2;$$

$$\textbf{(q4)} \quad q(1) = \mathbf{1}_{\mathcal{H}}.$$

It is known that at least in the case of the algebraic structures of the WWM phase space representation the mapping q is well defined [24].

In such way, this program will be indeed the quantization program if at least it is possible to construct in a sense uniquely the multiplication operations of Lie $\{\cdot,\cdot\}_{\mathrm{M}}$ (2.4) and Jordan $(\cdot\circ\cdot)$ (2.3) starting from a set of bilinear maps which are invariant under the action of the group $\mathcal{G}_{\mathrm{C}}$ of affine canonical transformations. When this test case will be verified then it will be possible to consider other basic subgroup of the affine transformations.

4. The basic group $\mathcal{G}_{\mathrm{C}}$: nonclassical algebraic structures

The purpose of this section is the construction of nonclassical Lie and Jordan operations in $\mathcal{A}$ on the basis of a set of the partial Moyal brackets $\left\{\{\cdot,\cdot\}^{(k)},\ \ k\geq 0\right\}$ which are the $\mathcal{G}_{\mathrm{C}}$-invariant bilinear operations. It is easy to see that for $k\geq 2$ there is no any $k=m$ such that $\{\cdot,\cdot\}^{(m)}$ is Lie or Jordan product. Hence, for a construction of nonclassical algebraic structures it is necessary to use an infinite linear combination of all $\left\{\{\cdot,\cdot\}^{(k)},\ \ k\geq 0\right\}$.

4.1. Lie structure

Let us consider a series $\{f,g\}_{\mathcal{G}_{\mathrm{C}}}$ with coefficients $c_{k,}$ $k=0,1,2,\cdots$;

$$\{f,g\}_{\mathcal{G}_{\mathrm{C}}}=\sum_{k=0}^{\infty}c_k\{f,g\}^{(k)}.$$

The operation $\{f,g\}_{\mathcal{G}_{\mathrm{C}}}$ will be a Lie product if

$$\{f,g\}_{\mathcal{G}_{\mathrm{C}}}=-\{g,f\}_{\mathcal{G}_{\mathrm{C}}} \tag{4.1}$$

and the Jacobi identity,

$$\left\{f,\{g,h\}_{\mathcal{G}_{\mathrm{C}}}\right\}_{\mathcal{G}_{\mathrm{C}}}+\left\{h,\{f,g\}_{\mathcal{G}_{\mathrm{C}}}\right\}_{\mathcal{G}_{\mathrm{C}}}+\left\{g,\{h,f\}_{\mathcal{G}_{\mathrm{C}}}\right\}_{\mathcal{G}_{\mathrm{C}}}=0 \tag{4.2}$$

is satisfied for all the observables.

Now we shall test the hypothesis that the conditions (4.1) and (4.2) determine the coefficients $\{c_k\}$ in a sense uniquely. First of all, it is evident that $\{f,g\}^{(k)}=(-1)^k\{g,f\}^{(k)}$ for all k and it follows that

$$\{f,g\}_{\mathcal{G}_{\mathrm{C}}}=\sum_{n=0}^{\infty}c_{2n+1}\{f,g\}^{(2n+1)}$$

Let us now consider the observables in $\mathcal{A}$ having the Fourier representation

$$f(x)\sim\int\tilde{f}(\alpha)\exp(i\alpha(x))d\alpha,$$

where $\alpha(x)$ is 1-form on M. It is easy to see that

$$\{\exp(i\alpha(x)),\exp(i\beta(x))\}_{\mathcal{G}_{\mathrm{C}}}=\sum_{n=0}^{\infty}c_{2n+1}\exp(i\alpha+i\beta)\left(-\{\alpha,\beta\}^{(1)}\right)^{2n+1} \tag{4.3}$$

Define a function $F(z)$ as formal power series,

$$F(z)=\sum_{n=0}^{\infty} c_{2n+1} z^{2n+1}, \qquad F(-z)=F(z), \tag{4.4}$$

where $\{c_{2n+1}\}$ are the coefficients to be found. Substituting equations (4.3) and (4.4) into the Jacobi identity (4.2), we derive the following functional equation,

$$F(z_1+z_2)F(z_3)+F(-z_2-z_3)F(z_1)+F(z_3-z_1)F(-z_2)=0 .$$

After formal manipulations we find that

$$F''(z)F'(z)-F'''(z)F(z)=0,$$

since $F(0)=F''(0)=0$. This equation for F under the relevant condition (4.4) immediately leads to

$$F''(z)=-d_1^2 F(z),$$

and then to

$$F(z)=d_2' \sin(d_1 z)=d_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(d_1)^{2n} z^{2n+1}, \tag{4.5}$$

where $d_2=d_2' d_1$. The obtained function $F(z)$ (4.5) determines all the coefficients c_{2n+1}, $n=0,1,2,\cdots$. Hence, the corresponding $\mathcal{G}_C$-invariant Lie product has the form

$$\{f,g\}_{\mathcal{G}_C}=d_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(d_1)^{2n}\{f,g\}^{(2n+1)} . \tag{4.6}$$

The coefficient d_2 is unessential for the bracket $\{\cdot,\cdot\}_{\mathcal{G}_C}$ as a Lie product. The operation (4.6) will be the same as the Moyal bracket (2.4) in the case that we put $d_2=1$ and $d_1=\hbar/2$.

4.2. Jordan structure

Now we consider the series $\sum_{k=0}^{\infty} c_k\{f,g\}^{(k)}$ once again and also we use arguments similar to the arguments given in the previous section with the purpose to construct a nonclassical Jordan product.

Let

$$(f\circ g)_{\mathcal{G}_C}=\sum_{k=0}^{\infty} c_k\{f,g\}^{(k)} .$$

The operation $(f\circ g)_{\mathcal{G}_C}$ will be Jordan product if

$$(f\circ g)_{\mathcal{G}_C}=(g\circ f)_{\mathcal{G}_C}, \tag{4.7}$$

and the Jordan identity,

$$\left((f\circ g)_{\mathcal{G}_C}\circ(f\circ f)_{\mathcal{G}_C}\right)_{\mathcal{G}_C}-\left(f\circ\left(g\circ(f\circ f)_{\mathcal{G}_C}\right)_{\mathcal{G}_C}\right)_{\mathcal{G}_C}=0, \tag{4.8}$$

is satisfied for all the observables [25].

We shall test now the hypothesis that the conditions (4.7) and (4.8) determine all the coefficients $\{c_k\}$ in a sense uniquely. First, it is easy to verify that the condition (4.7) leads to

$$(f \circ g)_{\mathcal{G}_C} = \sum_{n=0}^{\infty} \{f, g\}^{(2n)}.$$

Define function $G(z)$:

$$G(z) = \sum_{n=0}^{\infty} c_{2n} z^{2n}, \qquad G(-z) = G(z). \tag{4.9}$$

To be repeatedly applied the procedure that was already tested in the previous section here results in the following functional equation

$$G(z_1)G(z_2 + z_3) - G(z_1 + z_2)G(z_3) = 0. \tag{4.10}$$

After simple formal manipulations we find that the function $G(z)$ satisfies the equation

$$G''G' - G'''G = 0,$$

similar equation for the function $F(z)$ from the previous section but under the other condition (4.9). Last results in other form of the function $G(z)$:

$$G(z) = d_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} d_1^{2n} z^{2n}.$$

As a consequence, the $\mathcal{G}_C$-invariant Jordan product $(f \circ g)_{\mathcal{G}_C}$ takes the form

$$(f \circ g)_{\mathcal{G}_C} = d_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} d_1^{2n} \{f, g\}^{(2n)}. \tag{4.11}$$

It is directly visible that the structure of the Lie product (4.6) is similar to structure of the Jordan product (4.11) but the factors d_2 and d_2 formally in any way are not connected. That is, the real compatibility of the Lie and Jordan products taken into account it is necessary to add the appropriate condition [25]:

$$\left((f \circ g)_{\mathcal{G}_C} \circ h\right)_{\mathcal{G}_C} - \left(f \circ (g \circ h)_{\mathcal{G}_C}\right)_{\mathcal{G}_C} \sim \left\{g, \{f, h\}_{\mathcal{G}_C}\right\}_{\mathcal{G}_C}.$$

Now it is obvious that our program has resulted in the Section 4 in the algebraic operations which essentially coincide with the algebraic operations of the WWM phase space representation.

For a further consideration it should be kept in mind that the algebras may be complex.

5. Blokhintsev's conjecture and nonclassical algebraic structures

This section gives a little bit greater than one more example. Actually it demonstrates that it is convenient to get start for construction of nonclassical algebraic operations of Lie and Jordan, proceeding from (in a sense complete) a set of partial brackets rather then from a basic group. A set of partial brackets will be refered to as complete if on its basis it is possible to construct the algebraic operations having as a limiting (particular) case the appropriate classical operations of pointwise multiplication and Poisson bracket.

It is lot of years back Blokhintsev has proposed (however, both in the context and under the form distinct from ours) [13-15] to use structurally very simple sequence of the following bilinear operations,

$$\left\{ \frac{\partial^n f}{(\partial x^i)^n} \frac{\partial^n g}{(\partial x^j)^n} \right\}_{n=0}^{\infty} \quad \text{where } i \neq j.$$

We accept here the Blokhintsev proposition and take as starting point for construction of the nonclassical algebraic operations the following set of brackets on $\mathcal{A}$ (these brackets will be called the partial Blokhintsev brackets)

$$\left\{ \{f,g\}_{\mathrm{B}}^{(n)} = \begin{cases} \omega^{ij} \dfrac{\partial^n f}{(\partial x^i)^n} \dfrac{\partial^n g}{(\partial x^j)^n} & \text{if} \quad n = 2k+1 \\ \sigma^{ij} \dfrac{\partial^n f}{(\partial x^i)^n} \dfrac{\partial^n g}{(\partial x^j)^n} & \text{if} \quad n = 2k \end{cases} \right\}_{n=0}^{\infty} \tag{5.1}$$

The nonzero elements of σ^{ij} are $\sigma^{12} = \sigma^{21} = 1$. The lowest partial Blokhintsev brackets are correspondingly the pointwise multiplication and the Poisson bracket.

Certainly, (5.1) is a subset of the set of partial Moyal brackets and this circumstance can not be a surprise. Really, (5.1) can be characterized by the affine group of hyperbolic rotations, $\mathrm{h}(M) \otimes M$ (semidirect product) which is a subgroup of the affine group of canonical transformations (Let us recall that the hyperbolic rotations in M set transformations, $x_1' = \lambda x_1$, $x_2' = \lambda^{-1} x_2$, $\lambda \neq 0$; at a hyperbolic rotation the representing point slides on hyperbola $x_1' x_2' = x_1 x_2$.)

Further in this section the nonclassical Lie and Jordan algebraic structures are constructed on the basis of the set of partial Blokhintsev brackets (5.1). In summary of the section a general rule of constructing the nonclassical Lie and Jordan algebraic structures on the basis of an affine group is outlined.

5.1. Lie structure

Define a bilinear operation $\{f,g\}_{\mathrm{hb}}$ on $\mathcal{A}$ by linear combination of the odd order partial Blokhintsev brackets (5.1)

$$\{f,g\}_{\mathrm{hb}} = \sum_{k=0}^{\infty} c_{2k+1} \{f,g\}_{\mathrm{B}}^{(2k+1)} \tag{5.2}$$

It is necessary to note that for construction of Lie operation we have chosen from the very beginning an expression containing only odd order derivatives. If it looks completely naturally in the case of partial Moyal brackets then in the case under consideration the inducing reason of physical character was implicitly used. Namely, Lie operation in physics is connected to generation of time evolution of a system. If this evolution is reversible it must contain only partial brackets with odd order derivatives by momentum (the equations of motion are invariant under the transformation $(t,p) \leftrightarrow (-t,-p)$. This is a general remark and in any case it should be meant.

First, it is evident that the operation $\{f,g\}_{\rm hb}$ is antisymmetric since all the partial Blokhintsev brackets $\{f,g\}_{\rm B}^{(2k+1)}$ are antisymmetric (by definition),

$$\{f,g\}_{\rm hb}=-\{g,f\}_{\rm hb},$$

then it is necessary only to require the operation $\{f,g\}_{\rm hb}$ to satisfies the Jacobi identity,

$$\{f,\{g,h\}_{\rm hb}\}_{\rm hb}+\{h,\{f,g\}_{\rm hb}\}_{\rm hb}+\{g,\{h,f\}_{\rm hb}\}_{\rm hb}=0 \tag{5.3}$$

for all the observables.

Now we shall test the hypothesis that (5.3) determines all the coefficients $\{c_k\}$ in a sense uniquely by the same way as in the Section 4. In order to test the above hypothesis we shall use the same procedure as in Section 4.

First of all we have at hand the following expressions,

$$\left\{e^{i\alpha},e^{i\beta}\right\}_{\rm hb}=\sum_{k=0}^{\infty}c_{2k+1}\left\{e^{i\alpha},e^{i\beta}\right\}_{\rm B}^{(2k+1)}=\omega^{ij}\sum_{k=0}^{\infty}c_{2k+1}\left[-\left(\alpha_i\beta_j\right)^{2k+1}\right]e^{i\alpha+i\beta}$$

where α and β are linear functionals (on M).

$$\left\{e^{i\alpha},\left\{e^{i\beta},e^{i\gamma}\right\}_{\rm hb}\right\}_{\rm hb}=\omega^{ij}\sum_{k=0}^{\infty}c_{2k+1}\left[-\left(\alpha_i\beta_j\right)^{2k+1}\right]\left\{e^{i\gamma},e^{i\alpha+i\beta}\right\}_{\rm hb}$$

$$=\omega^{ij}\sum_{k=0}^{\infty}c_{2k+1}\left[-\left(\alpha_i\beta_j\right)^{2k+1}\right]\omega^{lm}\sum_{n=0}^{\infty}c_{2n+1}\left[-\left(\gamma_l\alpha_m+\gamma_l\beta_m\right)^{2n+1}\right]e^{i\alpha+i\beta+i\gamma}.$$

All the brackets in the Jacobi identity can be derived by the cyclic substitution of the triple (α,β,γ).

Define a function $F(z)$ as formal power series in z,

$$F(z)=\sum_{n=0}^{\infty}c_{2n+1}z^{2n+1},\qquad F(-z)=-F(z), \tag{5.4}$$

Under such a definition, the Jacobi identity can be rewritten as a functional equation for the function $F(z)$ or as formal power series in six variables,

$$\begin{aligned}&\left[F(z_1)-F(y_1)\right]\left[F(z_2+y_3)-F(y_2+z_3)\right]\\ &+\left[F(z_2)-F(y_2)\right]\left[F(z_3+y_1)-F(y_3+z_1)\right]\\ &\left[F(z_3)-F(y_3)\right]\left[F(z_1+y_2)-F(y_1+z_2)\right]=0.\end{aligned}$$

It is natural to set $F(z)\neq 0$.

Differentiating formally the last functional equation one time with respect to z_1 and twice with respect to y_3 and then setting $z_1=z_2=z,\ y_2=y_3=0$; we obtain the following equation,

$$F'(z)F''(z)-F(z)\left[F''(z)\right]'=0,$$

where conditions $F(0)=F''(0)=0$ have been used.

The simplest way to determine the coefficients $\{c_{2k+1}\}_{k=0}^{\infty}$ is to solve the above formal differential equation. Under this aim, let us rewrite it in another form

$\left[F(z)F''(z)\right]' = 0$, and immediately conclude that $F''(z) = -d_1^2 F(z)$, where d_1 is a number. Then, it is easy to find out the formal solution of the last equation under the relevant condition $F(-z) = -F(z)$. Explicitly,

$$F(z) = d_2 \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{(2k+1)!} d_1^{2k} z^{2k+1}.$$

Finally, it is obtained that

$$c_{2k+1} = d_2 \frac{(-1)^k}{(2k+1)!} d_1^{2k}, \quad k = 0, 1, \cdots.$$

Here d_1 and d_2 are some constants. In such a way, we have constructed the Blokhintsev's Lie algebraic brackets. It only still remains to verify that these brackets really satisfy the Jacobi identity. It is not a hard problem and we can justify the result using the same trick with the function F as in the proof above. The verification is omitted here.

5.2. Yet another product is possible to construct

Let us here ask the question: Is it possible to construct a bilinear product satisfies Jacobi identity and involves the even order partial brackets only? The answer to this question is "yes". But at the same time it is almost evident because we can in the relevant construct procedure directly follow the above scheme.

Formally, starting from the following definition of the auxiliary function F

$$F(z) = \sum_{k=1}^{\infty} c_{2k} z^{2k},$$

we derive the same equation for the function F as in the previous section, but now $F(0) = F'(0) = 0$, and it is natural to set that $F''(0) = 2c_2 \neq 0$. Differentiating formally the functional equation one time with respect to z_1 and twice with respect to y_3 and then setting $z_1 = z_2 = z$, $y_2 = y_3 = 0$; we obtain the following equation

$$F'(z)F''(z) - F(z)F'''(z) - F'(z)F''(0) = 0,$$

where conditions $F(0) = F'(0) = 0$ have been used. Let us rewrite this equation in the form

$$F'(z)\left[F''(z) - F''(0)\right] - F(z)\left[F''(z) - F''(0)\right]' = 0,$$

and then conclude that

$$\left[F''(z) - F''(0)\right] = -d_1^2 F(z),$$

where d_1 is a number. The solution of this equation, in a form of power series, is

$$F(z) = A\sin(d_1 z) + B\cos(d_1 z) + d_1^{-2} F''(0),$$

and then, under conditions $F(-z) = F(z)$, $F(0) = 0$, it leads to the final expression,

$$F(z) = d_1^{-2} F''(0)\left[1 - \cos(d_1 z)\right].$$

6. As a conclusion

Further, the following case may be considered with use: the group of affine homothetic transformations. As «use» – we mean the Blokhintsev bracket and the Klein principle. On the other hand, it is clear that the program is presented here in a germ form. For example, it looks very much attractive to construct nonclassical algebraic structures on the base of semisimple Lie algebras. Thus, we can use the corresponding structure constants λ^{ijk} as starting point for definition of the following set of the new partial brackets,

$$\{f,g;h\}_{\lambda}^{(n)} = \lambda^{i_1 j_1 k_1}\cdots\lambda^{i_n j_n k_n}\frac{\partial^n f}{\partial x^{i_1}\cdots\partial x^{i_n}}\frac{\partial^n g}{\partial x^{j_1}\cdots\partial x^{j_n}}\gamma_{k_1\cdots k_n}^{l_1\cdots l_m}\frac{\partial^m h}{\partial x^{l_1}\cdots\partial x^{l_m}}.$$

Under the restrictions for this presentation we are setting here this possibility as an intriguing remark only.

In this connection and not only with, it looks useful to consider a possible application of nonclassical algebraic structures before to develop the formalism in itself.

6.1. Nonlinear wave equations

The algebraic principle which leads to the correspondence between nonlinear wave equations and linear operators has been found by Lax in the case of the KdV equation [26]. First of all we translate Lax's method into the language of the Moyal bracket.

Let $L(x)$, $A(x) \in \mathcal{A}$ and let $L(x)$, $A(x)$ are polynomial in p, $x=(q,p)\in\mathbb{R}^2$. The Moyal bracket (4.6) (set for definiteness $d_2=1$, $d_2=\frac{1}{2}$) is a Lie algebra structure on $\mathcal{A}$. Define an integral curve in $\mathcal{A}$ by the equation

$$\partial_t L + \{A,L\}_{\mathrm{M}} = 0. \qquad (6.1)$$

In this case

$$\Lambda = \int_M L(x,t)W(x)dx = \text{constant},$$

as well as all

$$\Lambda = \int_M \overset{\circ k}{L}(x,t)W(x)dx = \text{constant}, \quad \text{k}=1,2,\cdots;$$

where $W(x)$ is a Wigner function and

$$\overset{\circ k}{L} \equiv \underbrace{L\circ L\circ\ldots\circ L}_{k}.$$

Consider the KdV equation as an example. Set

$$L = p^2 - u(q,t), \quad A = 4p^3 - 6pu(q,t).$$

In this case the equation (6.1) leads to a nonlinear wave equation - the KdV equation. Indeed, after simple calculations

$$-u_t + \{4p^3 - 6pu, p^2 - u\}_{\mathrm{M}} = -u_t - 4\{p^3, u\}_{\mathrm{M}} - 6\{pu, p^2\}_{\mathrm{M}} + 6\{pu, u\}_{\mathrm{M}}$$
$$= -u_t + 12u_q p^2 - u_{qqq} - 12u_q p^2 - 6uu_q$$
$$= 0 ,$$

we obtain

$$u_t + 6uu_q + u_{qqq} = 0$$

– the well-known KdV equation. Here the following notations are used: $u_t = \partial u / \partial t$, $u_q = \partial u / \partial q$. Thus it is an example of the way in which we can connect non-linear wave equations with the Lie-Moyal algebra of p-polynomials (in $\mathcal{A}$). Some other examples of this procedure can be easity given (see [27] for a comparison). But just before let us recall that some known nonlinear wave equations in contrast to the KdV equation have terms involving even-order partial derivatives, for example the Boussinesq equation. At first sight it seems as likelihood hypothesis that in this case even-order partial Moyal brackets may be successfully used. But it is a bad idea because even-order partial Moyal brackets destroy the Lie algebraic structure of the equation (6.1). In general, the correct question on using the Jordan algebraic structure (4.11) in context of nonlinear wave equations we remain here without answer. Fortunately, there is another way to solve the problem. It consists in considering the p-polynomials L and A involving even number of nontrivial coefficients. To clarify this point we consider a simple example. Set

$$L = -p^3 + \frac{3}{2} pu(q,t) + v(q,t), \quad A = p^2 - u(q,t).$$

Then, after calculations similar to used above, the equation (6.1) leads to the following set of equations

$$\frac{3}{2}u_t - 2v_t = 0,$$
$$v_t - \frac{1}{4}u_{qqq} - \frac{3}{2}u_q u = 0.$$

Excluding from these equations $v(q,t)$ we obtain

$$3u_{tt} + \left(u_{qqq} + 6u_q u\right)_q = 0. \tag{6.2}$$

The equation (6.2) is among the forms of the Boussinesq equation.

Almost evident the generalization is in considering the p-polynomials of a general form:

$$L = \sum_{n=0}^{N} u_n(q,t) p^n, \quad A = \sum_{k=0}^{K} v_k(q,t) p^k,$$

and the corresponding equations for the coefficients of these p-polynomials.

The next step should be in involving the scattering theory in this frame (for example, using the reworking of [28]).

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