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RELATIVISTIC PARTICLE IN A CONSTANT ELECTRIC FIELD (CHARACTERISTIC REPRESENTATION)

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The characteristic representation (CR) developed by the authors (see [1] and references therein) is used to study the problem about the moving of a relativistic spinless particle in a constant electric field E. *New solution which is the eigenfunction of square transverse momentum operator is found. It is shown that the eigenvalues of this operator depend on the metric of Mincowski space* M^2. *Namely, the eigenvalues are complex and discrete in the* $M^2_{(-)}$ *space and the eigenvalues are continious in the space* $M^2_{(+)}$. *Using this new solution the total charge and current of the particle in the constant electric field are calculated.*

1. Introduction

F. Zauter (1931) [2] was perhaps the first who have described the problem of moving a relativistic spinor particle in a constant electric field E. Till now an opinion exists that this problem is the argument against the application of one particle Klein-Fock-Gordon (KFG) equation in a strong electric field. It is relevant to remark that apart from the Sauter problem there exist the Klein paradox (1929) and the motion of relativistic particle in an external scalar field [1]. In all these problems there are unstable solutions increasing with time t. There are no the unstable increasing solutions for the moving particle in a constant magnetic field H. If the above problems are considered in so-called CR (or Goursat problem) then some regularity is observed. Namely, the Cauchy problem solutions which increase with the time correspond to the appearance of space-like solutions of Minkowski space $M^2_{(-)}$. We demonstrate this in detail.

From the CR viewpoint the free solution of Klein-Fock-Gordon (KFG) equation has the form [3]

$$\Psi(x_1,x_2,z,t)=\hat{S}(z,t)\left|in\right\rangle = J_0\left(\sqrt{(c^2t^2-z^2)(k_0^2-\Delta_\perp)}\right)\left|in\right\rangle , \tag{1}$$

where the evolution operator $\hat{S}$ should satisfy the Riemann condition along the characteristics $c|t| = |z|$

$$\hat{S}\Big|_{|t|=|z|/c}=1 ,$$

and $\left|in\right\rangle$ is the initial value of the wave function along the characteristics $c|t| = |z|$.

We recall that in the CR in contrast to the Cauchy problem only one value of wave function is given on the characteristics.

The operator $\Delta_\perp$ in Eq.(1) depends on the transverse coordinates

$$\Delta_\perp = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \quad \text{and } k_0 = \frac{\mu c}{\hbar}.$$

The operator $\Delta_\perp$ has the eigenfunction which can be either plane wave

$$\Delta_\perp[\exp i(k_1 x_1 + k_2 x_2)] = -k_\perp^2 \exp i(k_1 x_1 + k_2 x_2), \quad k_\perp^2 = k_1^2 + k_2^2,$$

or normalized MacDonald function, i.e., $\Delta_\perp K_0(k_\perp x_\perp) = k_\perp^2 K_0(k_\perp x_\perp)$ and

$$\int_0^\infty x\, \mathrm{d}x K_0^2(kx) = 1/2k^2.$$

The MacDonald function $K_m(kx)$ with $m \neq 0$ is not normalized because it has the strong singularity at $x = 0$.

We have shown earlier [3] that the fundamental solution of the wave equation in the $M^2_{(+)}$ space (where $c^2t^2 - z^2 \geq 0$) can be constructed from plane waves and the fundamental solution in the $M^2_{(-)}$ space (where $c^2t^2 - z^2 < 0$) can be constructed from MacDonald functions. The fundamental solution (1) has discontinuity on the characteristics $c|t| = |z|$ which separate $M^2_{(+)}$ space from $M^2_{(-)}$ space.

We recall that the free causal Feynmann propagator consists of the sum of two solutions as well as. One solution belongs to $M^4_{(+)}$ (or $M^2_{(+)}$). The other belongs to $M^4_{(-)}$ (or $M^2_{(-)}$).

The aim of the present paper is to consider the Sauter problem from CR viewpoint (or Goursat problem viewpoint) [1]. We show also that in a constant electric field the states are classified by eigenvalue of square transverse momentum operator i.e. operator $\Delta_\perp$ is the symmetry operator. The eigenvalues of this operator take discrete values in the $M^2_{(-)}$ space and continuous spectrum in the space $M^2_{(+)}$. We demonstrate that the moving of the particle in the constant electric field E is the transition from the initial bound state with discrete spectrum into the finite state with continuous spectrum of transverse momentum.

2. Charged particle in a constant electric field

Let us consider a particle with the charge e and the mass μ in the constant electric field E determined by potentials

$$A_z = -(E/2)ct, \quad A_0 = -(E/2)z. \tag{2}$$

We choose the axis z along the electric field.

KFG equation with the potential (2) has the form

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{c^2\partial t^2} + \Delta_\perp - k_0^2 + \frac{ieE}{\hbar c}\left(z\frac{\partial}{\partial ct} + ct\frac{\partial}{\partial z}\right) - \frac{e^2E^2}{4\hbar^2c^2}\left(c^2t^2 - z^2\right)\right)\Psi = 0. \tag{3}$$

Classification of states in Eq.(3) is over the eigenfunctions of the symmetry operator $\Delta_\perp$ because the variables x_1, x_2, are cyclic. From the CR viewpoint we seek the solution of Eq.(3) in the form

$$\Psi(x_1, x_2, z, t) = \hat{S}(z,t)|in\rangle,$$

where the evolution operator $\hat{S}(z,t)$ must be equal to 1 along the characteristics $|t| = |z|/c$ and the initial state $|in\rangle$ is the eigenfunction of the symmetry operator $\Delta_\perp$. These conditions are satisfied if the evolution operator $\hat{S}(z,t)$ is

$$\hat{S}(z,t) = \exp\left(\frac{i\alpha}{4}(c^2t^2 - z^2)\right)F\left(\hat{a},1,-\frac{i\alpha}{2}(c^2t^2 - z^2)\right), \quad (4)$$

here $\alpha = |e|E/\hbar c$ and $F(\hat{a},1,\tau)$ is degenerate hypergeometric function with operator parameter $\hat{a} = 1/2 - i/2k^2(k_0^2 - \Delta_\perp)$.We drop the second linear independent solution of Eq.(3) because it contains the factor $\ln(c^2t^2 - z^2)$. It diverges along characteristics $|t| = |z|/c$.

When $E \to 0$ then the evolution operator (4) tends to free evolution operator in the two-dimensional Minkowski space M^2 [3]

$$\hat{S}(z,t) \xrightarrow[E\to 0]{} J_0\left(\sqrt{(c^2t^2 - z^2)(k_0^2 - \Delta_\perp)}\right), \quad (5)$$

where J_0 -is Bessel function of first kind with index zero.

To prove Eq. (5) we use the definition of degenerate hypergeometric function and the asymptotic of Γ- function

$$\Gamma(x + n)/\Gamma(x) \to x^n \quad \text{at } |x| \to \infty.$$

In our papers those relating to the CR we have repeatedly indicated that from CR viewpoint we must consider two regions [1]: the region $M^2_{(+)}$, where $c^2t^2 - z^2 > 0$ (or $|t| > |z|/c$) and the region $M^2_{(-)}$ where $c^2t^2 - z^2 < 0$ (or $|t| < |z|/c$). We consider first the region $M^2_{(+)}$. As an initial state we take plane wave $\exp(ik_1x_1 + ik_2x_2)$, which is the eigenfunction of operators $-i\hbar\partial/\partial x_1$ and $-i\hbar\partial/\partial x_2$. On the states of plane waves the operator parameter $\hat{a}$ of degenerate hypergeometric function becomes the number: $\hat{a} \to a = 1/2 - i/2\alpha(k_\perp^2 + k_0^2)$, and the solution from CR viewpoint in space $M^2_{(+)}$ is

$$\Psi^{(+)}(x_1, x_2, z, t) = \text{const}\exp i(k_1x_1 + k_2x_2)S(z,t), \quad (6)$$

where the form of $S(z,t)$ (see Eq.(4))

$$S(z,t) = \exp\left(\frac{i\alpha\tau^2}{4}\right)F\left(a,1,-\frac{i\alpha\tau^2}{2}\right), \quad (7)$$

here $\tau^2 = c^2t^2 - z^2$ is the square of 2-D interwal of the space $M^2_{(+)}$.

The function (7) can be expanded over the eigenfunctions of 1-D wave operator [1]

$$\hat{L} = \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$$\exp\left(\frac{i\alpha\tau^2}{4}\right)F\left(a,1,-\frac{i\alpha\tau^2}{2}\right)=\int\limits_0^\infty J_0\left(\sqrt{(c^2t^2-z^2)Q^2}\right)\Phi(Q)Q\,\mathrm{d}Q,$$

where $\Phi(Q)=\exp\left(iQ^2/\alpha\right)F\left(a,1,-2iQ^2/\alpha\right)$.

Thus, this problem has a symmetry of the oscillator potential. The particle in the constant electric field can be regarded as the oscillator with the complex (imaginary) frequency.

The solution (6) do not lead to the new result. The solution in the plane wave form is well known. (See [4], page 131 and references therein).

Next, we consider the evolution operator $\hat{S}$ in the space $M^2_{(-)}$ where $z^2 - c^2t^2 > 0$, (or $|t| < |z|/c$). In this case we take the initial state $|in\rangle$ as the superposition of MacDonald functions K_0 [3]. The operator $\hat{a}$ becomes also the number and is equal to $\hat{a}\to a=1/2+i/2\alpha(k_\perp^2+k_0^2)$. Thus, the solution in space $M^2_{(-)}$ is

$$\Psi^{(-)}=\text{const }K_0(k_\perp x_\perp)\exp\left(\frac{-i\alpha\left(z^2-c^2t^2\right)}{4}\right)F\left(a,1,\frac{i\alpha}{2}(z^2-c^2t^2)\right). \quad (8)$$

As regards of MacDonald function K_0, it (as has been stated in the introduction) is the eigenfunction of the operator $\hat{p}_\perp^2=-\hbar^2\Delta_\perp$ because $\Delta_\perp K_0(k_\perp x_\perp)=k_\perp^2K_0(k_\perp x_\perp)$ (where $x_\perp=\sqrt{x_1^2+x_2^2}$) and it obeys to the normalizing integral

$$\int\limits_0^\infty x\,\mathrm{d}xK_0(kx)K_0(qx)=\frac{1}{2}\frac{\ln(k^2/q^2)}{(k^2-q^2)} \quad (9)$$

provided that

$$\mathrm{Re}(k+q)>0. \quad (10)$$

The condition (10) can be satisfied if we require that the parameter a of hypergeometric function in Eq.(8) is whole negative number that is $a=-n$ or

$$k_\perp^2=k_n^{(\pm)2}=k_0^2(1\pm i\gamma_n),\qquad \gamma_n=(E/E_s)(2n+1), \quad (11)$$

where $n = 0,1,2...$ and $E_s=\mu^2c^3/e\hbar$ is the constant Schwinger field.

Taking into account (11) we rewrite the normalizing integral of Eq.(9)

$$2\pi\int\limits_0^\infty x_\perp\,\mathrm{d}x_\perp K_0(k_n^{(+)}x_\perp)K_0(k_n^{(-)}x_\perp)=\pi\,\mathrm{arctg}\,\gamma_n/k_0^2\gamma_n, \quad (12)$$

where

$$k_n^{(\pm)}=k_0(1+\gamma_n^2)^{1/4}\exp(\pm(i/2)\,\mathrm{arctg}\,\gamma_n),\ \left|\arg k_n^{(\pm)}\right|\le\pi/4.$$

There are two linear independent solutions here

$$\Psi_{(\pm)}^{(-)}=\text{const }K_0(k_n^{(\pm)}x_\perp)S_n^{(\pm)}(z,t), \quad (13)$$

where

$$S_n^{(\pm)}(z,t)=\exp\left(\mp i\alpha/4(z^2-c^2t^2)\right)F\left(-n,1,\pm(i\alpha/2)(z^2-c^2t^2)\right). \quad (14)$$

These solutions can be associated with the different sign of the electric charge of the particle (antiparticle).

From Eq.(11) it follows that the value of transverse momentum is complex one and takes discrete values in the space $M^2_{(-)}$ (when $|t| < |z|/c$). It should be noted that the solution (13) is new. It was unknown earlier.

If in the integral (12) the strength $E \to 0$ then this expression is equal to 1 (in terms of $\frac{\pi}{k_0^2}$) and if $E \to \infty$ then wave functions of the particle (antiparticle) are not overlapped.

3. Physical statement of the problem

Let at $|t| < |z|/c$ there is a free particle in the constant electric field with charge $+e$ and with the discrete momentum $k_n^{(+)}$.

This state (according to (13)) is described by the wave function

$$\Psi^{(-)}_{(+)} = B_n K_0\left(k_n^{(+)} x_\perp\right) S_n^{(+)}(z,t), \tag{15}$$

where the form of $S_n^{(+)}$ was taken from Eq.(14) and the constant B_n is the normalizing constant determined by integral (12). When $|t| > |z|/c$ there is the state of the particle with momentum $\vec{k}(k_1,k_2)$ which is described by superposition of the plane waves

$$\Psi^{(+)} = \int \mathrm{d}k_1\, \mathrm{d}k_2 c_n(k_1,k_2) \exp i(k_1x_1 + k_2x_2) S(k_\perp, z, t) \tag{16}$$

where we take the form $S(k_\perp, z, t)$ from Eq. (7).

We recall that from the CR viewpoint the only continuity condition of wave function is required along the characteristics $|t| = |z|/c$, therefore

$$B_n K_0(k_n^{(+)} x_\perp) = \int \mathrm{d}k_1\, \mathrm{d}k_2 c_n(k_1,k_2) \exp i(k_1x_1 + k_2x_2)\,. \tag{17}$$

Here $c_n(k_1,k_2)$ is the amplitude of transition from the initial state of the space $M^2_{(-)}$ into the finite state of the space $M^2_{(+)}$,

$$c_n(k_1k_2) = (1/2\pi)B_n/(K^2 - ik_0^2\gamma_n), \quad K^2 = k_\perp^2 + k_0^2$$

and, respectively

$$|c_n(k_1,k_2)|^2 = \frac{1}{4\pi^2}\frac{|B_n|^2}{(K^4 + k_0^4\gamma_n^2)}\,. \tag{18}$$

Integrating Eq.(18) over transverse momentum we obtain

$$\int \mathrm{d}k_1\, \mathrm{d}k_2 |c_n(k_1,k_2)|^2 = (1/4\pi^2)\pi|B_n|^2 \operatorname{arctg}\gamma_n / k_0^2\gamma_n = 1/4\pi^2, \tag{19}$$

since from the normalizing integral (12) it follows that $\pi|B_n|^2 \operatorname{arctg}\gamma_n / k_0^2\gamma_n = 1$, therefore the transition amplitude $2\pi c_n(k_1,k_2)$ is the probability amplitude and the value $|2\pi c_n(k_1,k_2)|^2$ is the density of the transition probability. The total transition probability is equal to 1. This result is not a surprise, it follows at once from Eq.(17) using Purceval equality.

Taking into account the solution (16) with $c_n(k_1,k_2)$ determined by (18) let us calculate the total charge Q of the particle and the total current J. It can be shown that the current component J_z is different from 0 only

$$Q = \frac{ie}{c} \int d x_1 \, d x_2 \int_{-c|t|}^{c|t|} d z \left(\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t} \right), \tag{20}$$

and

$$J_z = -2e^2 / \hbar \int d x_1 \, d x_2 \int_{-c|t|}^{-c|t|} d z \, A_z |\Psi|^2 . \tag{21}$$

In Eqs.(20),(21) the odd terms with respect to z are omitted because after integration they are equal to zero. Let us note that integration limits (20),(21) for the variable z are finite. This variable satisfies the inequality $|z| \le c|t|, or - c|t| \le z \le c|t|$.

We perform the calculations when $\alpha(c^2t^2 - z^2) >> 1$, and use the asymptotic of hypergeometric function

$$F(a,1,\tau) \sim 1/\Gamma(a)(\tau)^{-(1-a)}\exp\tau, \quad |\arg\tau| \le \pi/2.$$

Substituting Eq.(16) into Eq.(20) we get

$$Q = e_0 \varepsilon(t) \left\{ 1 + \frac{i}{2 \operatorname{arctg} \gamma_n} \left(E_i \left(\frac{-\pi E_s (1 - i\gamma_n)}{E} \right) - E_i \left(\frac{-\pi E_s (1 + i\gamma_n)}{E} \right) \right) \right\}, \quad J_z = cQ , \tag{22}$$

where $\varepsilon(t) = |t|/t$ is sign function, $e_0 = eZ_1$ is the true renormalizing charge of particle including the infinitesimal logarithmic factor $Z_1 = \int_{-1}^{1} d u / 1 - u^2$. $E_i(\tau)$ is the integral exponential function with the asymptotic $E_i(z) \sim e^z/z$ if $|z| >> 1$.

In weak external field ($E/E_s << 1$) the total charge Q is $e_0\varepsilon(t)$, and in the strong field ($E/E_s >> 1$) the recharge of the particle takes place

$$Q = -\frac{2}{\pi} e_0 \varepsilon(t) S_i (\pi(2n+1)), \quad S_i(z) = \int_0^z d u \sin u / u . \tag{23}$$

The changing or the deformation of the total charge by an external field was predicted many years ago (Dirac (1934) and others). This phenomenon is called the vacuum polarization [5].

It is difficult to compare our results with Schwinger's [5] and other results because we use another initial states. This is similar to comparing the two energy values with respect to different energy levels.

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1. *A.A. Borghardt, D.Ya. Karpenko*, JETP **85**, 4, 635 (1997).
2. *F. Zauter*, Zs. für Phys. **69**, 11-12, 742 (1931).
3. *A.A. Borghardt, D.Ya. Karpenko*, J. Nonlinear Math. Phys. **5**, 4, 357 (1998)
4. *A.A. Grib, S.G. Mamaev, V.M. Mostepanenko*, Quantum Effects in Intense External Fields, Atomizdat, Moscow (1980) (in Russian).
5. *J. Schwinger*, Phys. Rev. **82**, 664 (1951).