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WAVE PACKETS OF TURBULENT TYPE IN NONLINEAR BOUNDARY PROBLEMS OF QUANTUM MECHANICS

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An initial boundary value problem for the linear Schrödinger equation with nonlinear functional boundary conditions is considered. It is shown that an attractor of the problem contains periodic piecewise constant functions on the complex plane with a finite number of the points of discontinuities on a period. The method of reduction of the problem to a system of integro-difference equations has been applied. Applications to optical resonators with feedback have been considered. The elements of the attractor can be interpreted as white and black solitons in nonlinear optics.

Keywords: Schrödinger equation, functional two points boundary conditions, asymptotic periodic piecewise constant distributions of relaxation type

Introduction

In this paper, an initial boundary value problem (IBVP) is considered, which describes dynamics of two free particles characterized by opposite impulses that are placed into a quantum box. Thus, we analyze the dynamics of a kicked charged particle moving in a double-well or a more complex potential which is placed at flat walls of the box. In [1], a deterministic version of classical Langevin problem has been studied, where the movement of a charged particle in a double-well potential is analyzed. It is shown that the Langevin problem can be reduced to the study of a family of iterated function systems containing a complex logistic map. This result provides physical meaning for the Julia set. Similar approach [2] is applied to the study of the initial value boundary problem for the Liouville equation with nonlinear dynamic boundary conditions. The problem describes a velocity of time evolution of the probability of particles at walls that confine the particles. Note that these velocities are nonlinear functions of the density of the probability of particles that occupy flat walls. The attractor of the problem has been constructed. This attractor contains periodic piecewise constant functions with finite, countable or uncountable (homeomorphic to the Cantor set) lines of discontinuities on a period, which propagate along the characteristics of the Liouville equation. Such elements of the attractor are called the limit generalized distributions of relaxation of pre-turbulent and turbulent type, with respect to the Sharkovsky classification [3]. In the present paper, we expand the results reported in [1,2] to the IBVP, where the motion of free particles characterized by different impulses is described by the generalised Shrödinger-type equations. The related operators are linear with small parameters, and with symbols that are polynomial functions

$$P_n(p) = \sum_{j=0}^{n} a_j p^j, \quad n = 1, 2, 3, \dots . \tag{1}$$

Here, $p \in R$, p corresponds to the operator $\hat{p} = -ih\,\mathrm{d}/\mathrm{d}x$, where $h > 0$ is a small parameter. If $n = 2$ then we deal with the Shrödinger equation. Let us define

$$\hat{E} = -ih\frac{\partial}{\partial t}, \quad \hat{p} = -ih\frac{\mathrm{d}}{\mathrm{d}x} \tag{2}$$

and consider the uncoupled system of equations

$$\left(-\hat{E} + P_n^1(\hat{p})\right) y_k(x,t) = 0, \quad k = 1, 2. \tag{3}$$

Let initial conditions to be of a special form

$$y_k(x,0) = \exp\left(\frac{i}{h}\alpha_k x\right). \tag{4}$$

Then we can find a solution in the form

$$y_1(x,t) = \exp\left(\frac{i}{h}\lambda_1^1 x + \lambda_1^2 t\right), \quad y_2(x,t) = \exp\left(\frac{i}{h}\lambda_2^1 x + \lambda_2^2 t\right) \tag{5}$$

where $\lambda_i^j \in R$, $i, j = 1, 2$.

The corresponding initial problem has been solved in [4], where it is shown that the problem may be reduced to the Hamilton–Jacobi equations

$$\lambda_2^1 + P_n^1\left(\lambda_1^1\right) = 0, \quad \lambda_2^2 + P_n^1\left(\lambda_1^2\right) = 0 \tag{6}$$

and to a system of transport equations, respectively:

$$\frac{\partial \varphi_1}{\partial t} + \frac{\partial P_n^1\left(\lambda_1^1\right)}{\partial p}\frac{\partial \varphi_1}{\partial x} = 0, \tag{7}$$

$$\frac{\partial \varphi_2}{\partial t} + \frac{\partial P_n^2\left(\lambda_2^1\right)}{\partial p}\frac{\partial \varphi_2}{\partial x} = 0. \tag{8}$$

Let us define

$$\frac{\partial P_n^1\left(\lambda_1^1\right)}{\partial p} = \lambda_1, \quad \frac{\partial P_n^2\left(\lambda_2^1\right)}{\partial p} = \lambda_2. \tag{9}$$

Then

$$\frac{\partial \varphi_1}{\partial t} + \lambda_1 \frac{\partial \varphi_1}{\partial x} = 0\,, \qquad (10)$$

$$\frac{\partial \varphi_2}{\partial t} + \lambda_2 \frac{\partial \varphi_2}{\partial x} = 0 \qquad (11)$$

where we assume that $\lambda_1 > 0$ and $\lambda_2 < 0$.

Now we consider functional boundary conditions

$$\varphi_1(0,t) = \varphi_2(0,t), \quad \varphi_2(l,t) = \Phi\left(\varphi_1(l,t)\right). \qquad (12)$$

Integration of these ODEs along the characteristics with account of boundary conditions (12) results in the relations:

$$\varphi_1(l,t) = \varphi_1(0, t - l/\lambda_1) = \varphi_2(0, t - l/\lambda_1) =$$

$$= \varphi_1(l, t - l/\lambda_1 - l/\lambda_2) = \Phi\left(\varphi_1(l, t - l/\lambda_1) - 1/\lambda_2\right). \qquad (13)$$

Define $\Delta = l/\lambda_1 + l/\lambda_2$. Then it follows from (13) that

$$\varphi_1(l,t) = \Phi\left(\varphi_1(l, t-\Delta)\right). \qquad (14)$$

The solutions of (14) can be found by step-by-step iteration of the initial function $h_1(t)$ over $[-\Delta, 0)$. Let us define $y(t) = \varphi_1(l,t)$. Then $h_1(t)$ can be determined by the method of characteristics, so that $y(t) = \varphi(t) = \varphi_1(t)$ for $t \in [-1/\lambda_2, 0)$ and $\varphi(t) = \varphi_2(t) = \varphi_1(t)$, $t \in [0, 1/\lambda_1)$ (see [5], Fig. 85).

1. Hamilton–Jacobi equations

The Hamilton–Jacobi equations have solutions

$$\lambda_1^1 = \alpha_1,\ \lambda_1^2 = -P_n^1(\alpha_2), \quad \lambda_2^1 = \alpha_1,\ \lambda_2^2 = -P_n^2(\alpha_2). \qquad (15)$$

Thus

$$S_k(x,t) = \alpha_k x - P_n^k(\alpha_k)t, \quad k = 1,2\,. \qquad (16)$$

For the solvability of the IBVP we should assume that

$$\frac{\partial P_n^1\left(\lambda_1^1\right)}{\partial p} < 0\,. \qquad (17)$$

Now for the Hamilton–Jacobi equations,

$$\frac{\partial S_k}{\partial t} + P_n^k\left(\frac{\partial S_1}{\partial t}\right) = 0\,, \quad k = 1,2\,, \qquad (18)$$

we postulate the periodic boundary conditions

$$S_1(0,t) = S_1(l,t), \ \ S_2(0,t) = S_2(l,t) \quad t > 0\,, \qquad (19)$$

$$S_1(x,0) = S_1^0(x), \quad S_2(x,0) = S_1^l(x), \quad 0 < x < l. \tag{20}$$

Next, we extended the initial conditions to $x \in R$ l-periodically. In this case, solutions of the initial problem for a phase will be a solution of the boundary problem with the aid of periodical extention of the linear phases $S_1(\zeta)$, $S_1(\eta)$, where

$$S_k(x,t) = \alpha_1 x - P_n^k(\alpha_1)t, \quad k = 1,2, \quad n = 0, 1, \dots . \tag{21}$$

2. WKB-approximation with a complex phase

Thus, we consider IBVP for two linear PDEs with the symbols, which are polynomial of order n = 2,3,… [6] and with nonlinear functional or dynamic boundary conditions. For example, for n = 2 we have two uncoupled Shrödinger equations. The boundary conditions represent the relations between amplitudes and phases of (in) and (out) waves at the walls of the quantum box. We consider 1D case, but the results may be generalized on 3D case. It should be noted that the boundary conditions include a phase-dependent exponential factor. The initial conditions have the form

$$u(x,t,h) = A(\omega)\left[\varphi_0(x,t)e^{i\omega S_1(x,t)}e^{-i\omega S_2(x,t)} + O(1/\omega)\right] \tag{22}$$

where $S_1, S_2 \geq 0$, φ_0 are smooth functions. If

$$S(x,t) = S_1(x,t) + iS_1(x,t), \quad \omega = 1/h \tag{23}$$

then a solution of (22) has the form

$$u(x,t,h) = A(1/h)\left(\varphi_0(x,t)e^{\frac{i}{h}S(x,t)} + O(h)\right). \tag{24}$$

The solutions are called WKB-solutions. Here, $h > 0$ is a small parameter. It means that we consider high-frequency approximation or approximation of geometric optics, that is called sometimes the approximation of thin laser beams. That is, for each fixed $t > 0$, a solution is «localized» in the vicinity of some curve (see [7, p. 33]). The motivation of the introduction of a small parameter $h > 0$ or «inner Planck constant» is that asymptotic solutions of this equations ($h \to 0$) are used for the quantization ([7, p. 31]). The construction of the asymptotic solutions can be provided by the method of reduction of the problem of a system of equations of quantum mechanics to a system of equations of classic mechanics: to the Hamilton–Jacobi equations for phases and transport equations or Liouville equations for amplitudes.

The construction of complex solutions of these equations of quantum mechanics for infinitely thin laser beams allows writing WKB-solutions in the form

$$u_k(x,t,h) = e^{iS_k/h}\sum_{j=0}^{m}\varphi_j^k(x,t)h^j, \quad k = 1,2 \tag{25}$$

where S_k and $\varphi_j^k(x,t)$ are solutions of the Hamilton–Jacobi equations and Liouville equations.

Note that the Hamilton-Jacobi equations can be solved exactly. The zero approximation can be determined with accuracy $O(h^2)$ and one is a real function, but another functions admit imaginary corrections to a phase, that is for each next π/2.

In the present paper, this method of reduction is applied to the boundary problems of quantum mechanics. The results can be transferred to problems of nonlinear optics, to the Ginzburg–Landau equations for a two-component order parameter: an example is the system of the Gor'kov equations, which describes the density of Cooper pairs in superconductors of type 2 [9], and so on.

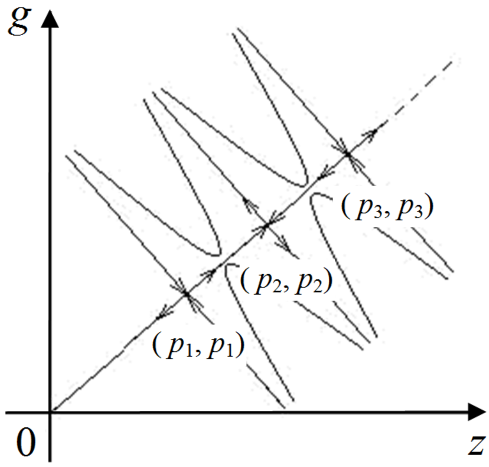

Fig. 1. The trajectories of hyperbolic dynamical systems with attractive and saddle points in a plane

$$u^1(0,t)=\lambda_1 u^2(0,t), \quad u^2(l,t)=\lambda_2 u^2(l,t), \quad t>0 . \tag{26}$$

Let us consider a system of partial differential equations with constant coefficients which have polynomial symbols

$$P_n(p)=\sum_{j=0}^{n} a_j p^j \tag{27}$$

where $P_n(p)$ is a polynomial of variable $p \in R^1$ of power $n = 1,2,\ldots$. Formally, the transformation of variable p into operator $\hat{p} = -h\,\mathrm{d}/\mathrm{d}x$ results in the differential operator (27):

$$P_n(\hat{p})=P_n\left(-ih\frac{\mathrm{d}}{\mathrm{d}x}\right)=\sum_{j=0}^{n} a_j(-ih)^j \frac{\mathrm{d}^j}{(\mathrm{d}x)^j} \tag{28}$$

with constant coefficients.

$$\left(-\hat{E}_k+P_n^k(\hat{p})\right)\phi(x,t)=0, \quad k=1,2 \tag{29}$$

where $\hat{E} = ih\,\partial/\partial t$.

3. Beck's type boundary conditions

We consider the functional boundary conditions

$$\phi_1(0,t,h)=S_1\left(\phi_2(0,t,h)\right), \quad \phi_2(0,t,h)=S_2\left(\phi_2(0,t,h)\right) \tag{30}$$

and the initial conditions

$$\phi_1(x,0,h) = h_1(x,h), \quad \phi_2(x,0,h) = h_2(x,h) \tag{31}$$

where $S_1, S_2 : R \to R$ are given functions. As follows from [1], such boundary conditions can describe the dynamics of a kicked free particle moving in quantum box with a double-well surface potentials. The related classical case is considered in [9]. Indeed, as noted by Beck [1], «Though we will usually call the dynamical variable in our equations the velocity of a particle», our approach is much more general. Double-well potentials have many applications in physics, in subject areas as diverse as chemical kinetics, non-equilibrium thermodynamics, elementary particle physics and cosmology.

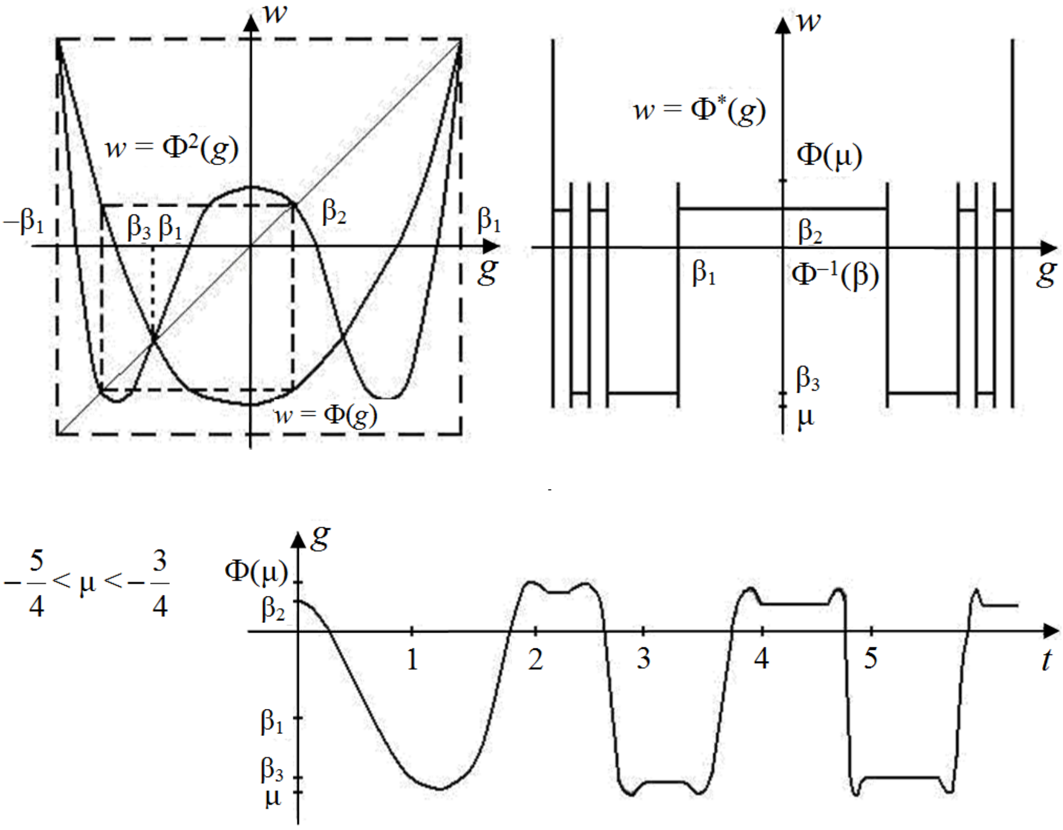

Fig. 2. Limit solutions of relaxation type

Really, at time *t*, a free particle gets a strength $c = a + ib$ in *x*-direction. Consider the velocity $v^-(t) = \left(u^-(t), w^-(t)\right)$ and $v^+(t) = \left(u^+(t), w^+(t)\right)$ before and after the kick. Then we have

$$u^+ = u^- + a, \quad w^+ = w^- + b \tag{32}$$

that is equivalent to

$$z^+ = z^- + c \tag{33}$$

where $c = a + ib$ is a complex constant. Next, we assume that the strength is acting at each of two flat walls of the quantum box. Then we can consider a generalization of (33) in a nonlinear case so that

$$\psi_1(0,t) = \Phi(\psi_2(0,t)), \quad \psi_2(l,t) = \Phi(\psi_1(l,t)) \tag{34}$$

where $\Phi : I \to I$ is a given function. I is an open bounded interval. Here, $\Psi_1 = z^+$ and $\Psi_1 = z^-$. The index (±) labels quantities before (–) and after (+) the kick. If Φ:= = *Id*, where *Id* is an identical map, we obtain linear boundary conditions of type (33).

It is shown in [1] that in an unbounded homogeneous space, complex nonlinear mappings Φ arise as stroboscopic mappings of certain classical particle dynamics. In a sense, that is the deterministic version of a typical Langevin problem. Generalization of [1] is considered in [2] by the example of 2D initial boundary value problem for the Liouville equation with nonlinear dynamic boundary conditions which describes the velocity of time evolution of the probability of particles at the walls that confine the particles. These velocities are nonlinear functions of the density of the probability of occupation of the flat walls. The attractor of the problem has been constructed. This attractor contains periodic piecewise constant functions with finite, countable or uncountable points of discontinuities per a period, which propagates along the characteristics of the Liouville equation. Such elements of the attractor are called distributions of relaxation of pre-turbulent and turbulent type, respectively. There are also random distributions of particles, which can be produced by the nonlinear feedback at the walls. The results has been obtained by the reduction of the problem to dynamical system which is described by system of difference equations depending on coordinates and momenta. It is shown that a change in these parameters results in period-doubling bifurcations of elements of the attractor on a 4-dimensional torus. The problem is solved in the class of quasi-periodic functions.

The main contribution to the behaviour of solutions of IBVP is made by boundary conditions in complex space because we assume that the equations of quantum mechanics are linear. Next, these equations can be reduced in WKB-approximation to a canonical system, which represents coupled system of the Hamilton–Jacobi and transport equations for the phases and amplitudes, respectively. The problem is reduction of the boundary conditions for the quantum equation to the boundary conditions for classical canonical equations. A similar example has been done in [1] for the problem which describes the dynamics of a charged particle moving in some arbitrary potential and magnetic field under the influence of kicks.

4. Decomposition on amplitudes and phases

In this section, the problem of decomposition of density u is discussed:

$$u = u_1 e^{i\tau S_1} + \ldots + u_k e^{i\tau S_k} \tag{35}$$

where $\tau = 1/h$. Phases S_j should be found. If $j = 1$ then we have the known WKB-decomposition. If there is a unique term in the series then only the phase factor appears. But for many terms, the choice of relative phases for the Cauchy problem is important. Special procedure for the determination of phases is presented below.

We begin with the Cauchy problem. Let the solution be

$$u(x,t)=\sum_{l=0}^{\infty} h^l \phi_l(x,t). \tag{36}$$

As an example, we consider the initial problem

$$\left(\hat{E}+\hat{P}_n\right)\psi(x,t)=0, \tag{37}$$

$$\psi(x,0)=\mathrm{e}^{ih(\lambda_1 x+\lambda_2 t)}\phi_0 x. \tag{38}$$

The solutions are

$$y(x,t)=\mathrm{e}^{ih(\lambda_1 x+\lambda_2 t)}\phi(x,t). \tag{39}$$

Substituting (39) into equation

$$\left[-ih\frac{\partial}{\partial t}+P_n\left(-ih\frac{\partial}{\partial x}\right)\right]y(x,t)=0, \tag{40}$$

we get

$$\left[\left(\lambda_2-ih\frac{\partial}{\partial t}\right)+P_n\left(\lambda_1-ih\frac{\partial}{\partial x}\right)\right]\phi(x,t)=0. \tag{41}$$

The initial conditions are

$$y(x,0)=\mathrm{e}^{\frac{i}{h}\lambda_1 x}\phi(x,0)=\mathrm{e}^{\frac{i}{h}\alpha x}\phi_0(x). \tag{42}$$

It follows from (42) that

$$\lambda_1=\alpha, \quad \phi(x,0)=\phi_0(x). \tag{43}$$

Now we not that the term

$$\left(\lambda_2-ih\frac{\partial}{\partial t}\right)+P_n\left(\lambda_1-ih\frac{\partial}{\partial x}\right) \tag{44}$$

may is obtained from the function

$$\left(\lambda_2-ihE'\right)+P_n\left(\lambda_1-ihp'\right) \tag{45}$$

by formal transformation $E'\to\partial/\partial x$. Now we expand functional (45) into a Taylor series in terms of h so that

$$F(h)=\sum_{k=0}^{n}\frac{h^k}{k!}\frac{\mathrm{d}^k}{\mathrm{d}h^k}F(h)\bigg|_{h=0}=\left(\lambda_2+P_n(\lambda_1)\right)+$$

$$+\ h\left(-iE'-\frac{\partial P_n}{\partial p}(\lambda_1)\,p'\right)+\sum_{k=2}^{n}\frac{h^k}{\partial p^k}(-i)^k\frac{\partial^k P_n}{\partial p^k}(\lambda_1)(P')^k. \tag{46}$$

After transformations $E' \to \dfrac{\partial}{\partial t}$ and $p' \to \dfrac{\partial}{\partial x}$

$$\left(\lambda_2 - ih\frac{\partial}{\partial t} + P_n\left(\lambda_1 ih\frac{\partial}{\partial x}\right)\right) = \left(\lambda_2 + p_n\left(\lambda_1\right)\right) -$$

$$- ih\left(\frac{\partial}{\partial t} + \frac{\partial P_n}{\partial p}\left(\lambda_1\right)\frac{\partial}{\partial x}\right) + \sum_{k=2}^{n}\frac{\left(-ih\right)^k}{k!}\frac{\partial^k P_n}{\partial p^k}\left(\lambda_1\right)\frac{\partial^k}{\partial x^k}. \qquad (47)$$

Relation (47) can be rewritten as

$$\left(\lambda_2 + P_n\left(\lambda_1\right)\right)\phi\left(x,t\right) - ih\left(\frac{\partial \phi}{\partial t} + \frac{\partial P_n}{\partial p}\left(\lambda_1\right)\frac{\partial \phi\left(x,t\right)}{\partial x}\right) +$$

$$+ \sum_{k=2}^{n}\frac{\left(-ih\right)^k}{k!}\frac{\partial^k P_n}{\partial p^k}\left(\lambda_1\right)\frac{\partial^k \phi\left(x,t\right)}{\partial x^k} = 0. \qquad (48)$$

The exact solution of the problem cannot be derived from (47) by the choice of constants λ_1, λ_2 and function $\phi\left(x,t\right)$. By setting additional terms in expansion (48) equal to zero, we can get asymptotic solution on h $(h \to 0)$.

Thus we have the equation

$$\lambda_2 + P_n\left(\lambda_1\right) = 0 \qquad (49)$$

and

$$\frac{\partial \phi}{\partial t} + \frac{\partial P}{\partial p}\left(\lambda_1\right) + \frac{\partial \phi}{\partial x} = 0. \qquad (50)$$

Equation (49) has a solution

$$S\left(x,t\right) = \alpha x - P_n\left(\alpha\right)t,\ \lambda_1 = \alpha,\ \lambda_2 = -P_n\left(\alpha\right). \qquad (51)$$

Equation (49) can be rewritten in more clear form which is equivalent to (50). Indeed, consider a vector field v at plane $(x;t)$ with coordinates which are independent of x; t. The vector field of this type is

$$v = \left(\frac{\partial P_n^k}{\partial p_k}\left(\lambda_1\right), 1\right), \quad k = 1,2. \qquad (52)$$

It means that there exists a derivative along the trajectories of vector field v in the left-hand part of (50). Then the transport equation is an ODE

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = 0 \qquad (53)$$

where d/dt is a derivative along the trajectories of the vector field. It follows from (53) that ϕ must be constant along the trajectories. The transport equation allows obtaining of a solution with accuracy $O(h^2)$. To obtain the succeeding terms of the asymptotic series, we must find $\phi(x,t)$ as a formal power series in h. For ϕ_0, we obtain the transport equation again. Then the right-hand part of ODE is of order $O(h^2)$ for each integer $s > 0$.

5. Complex transport equations in a first approximation

Next, we consider a function $\phi_1(x,t)$. Then, with acuracy of order $O(h^2)$, we get the equation

$$\frac{\partial\phi_1}{\partial t}+\frac{\partial P_n}{\partial p}(\lambda_1)\frac{\partial\phi_1}{\partial x}=-\frac{i}{2}\frac{\partial^2 P_n}{\partial p^2}(\lambda_1)\frac{\partial^2\phi_0}{\partial x^2}, \tag{54}$$

which can be written as

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}t}=-\frac{i}{2}\frac{\partial^2 P_n}{\partial p^2}(\lambda_1)\frac{\partial^2\phi_0}{\partial x^2} \tag{55}$$

by the definition of d/dt. If $\phi_0(x;t)$ has been determined earlier, the integration of (55) allows obtaining $\phi_1(x;t)$.

Further, we consider the terms of the equation that are of h^3, h^4, ... by order. Then we obtain a recurrent system of equations, which determine functions $\phi_s(x;t)$. Each successive function can be obtained from the previous one by integration along vector field v.

6. Systems of linear quantum equations with nonlinear boundary conditions

Consider the following system of equations

$$-h\frac{\partial\psi_k}{\partial t}+H_k(x,p,t)\psi_k=0, \quad k=1,2, \tag{56}$$

with the initial conditions

$$\psi_k(x,0)=\mathrm{e}^{i/h}S_0^k(x)\psi_0(x), \tag{57}$$

and the boundary conditions

$$\psi_1\bar{\psi}_1=\Phi_1(\psi_2\bar{\psi}_2)\big|_{x=0}, \quad \psi_2\bar{\psi}_2=\Phi_2(\psi_1\bar{\psi}_1)\big|_{x=1} \tag{58}$$

where Φ_1, Φ_2 are assigned functions. Here $\bar{\psi}$ is the quantity conjugated to ψ. The Hamiltonian of the problem $H_k(x; p; t)$ satisfies the estimation

$$\left|D_x^\alpha D_p^\beta H_k(x,p,t)\right|\le C_{\alpha,\beta}\left(1+|x|+|p|\right)^m \tag{59}$$

where $m > 0$ is a fixed number, α, β are multi-indexes, $C_{\alpha\beta}$ are constants.

If $H_k = P_n^{(k)}$ then the problem can be reduced to a system of equations with accuracy $O(h^2)$:

$$\frac{\partial\phi_1^0}{\partial t}+\frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial\phi_1^0}{\partial x}=0\,, \tag{60}$$

$$\frac{\partial\phi_2^0}{\partial t}+\frac{\partial P_n^2}{\partial p}(\lambda_2)\frac{\partial\phi_2^0}{\partial x}=0\,, \tag{61}$$

with the boundary conditions

$$\left|\phi_1^0\right|^2=\Phi_1\left(\left|\phi_2^0\right|^2\right)\Bigg|_{x=0}, \quad \left|\phi_2^0\right|^2=\Phi_2\left(\left|\phi_1^0\right|^2\right)\Bigg|_{x=1} \tag{62}$$

and the initial conditions

$$\phi_k^0(x,0)=h_k(x), \quad k=1,2\,. \tag{63}$$

A solution has the form

$$u_1^0(x,t)=y(t-x/\lambda_1), \quad u_2^0(x,t)=y(t+x/\lambda_2) \tag{64}$$

where $\lambda_{1,2}\to\partial P_n^{1,2}\big/\partial p(\lambda_{1,2})$, $n=0,1,2,...$ are coefficients of the related hyperbolic equations. Assume that $\lambda_1\lambda_2<0$. Then we get (see [9,5]):

$$y(t+2\Delta)=\Phi(y(t)), \quad t\in[-1,\infty), \quad \Delta=l/V_1+l/V_2 \tag{65}$$

accompanied by the initial condition

$$y(t)\big|_{[-1,1)}=h(t)\,, \tag{66}$$

where $h(t)=\phi_1(-t)$ at $t\in[-1,0)$ and $h(t)=\phi_2(-t)$ at $t\in[0,1)$. A difference equation can be obtained by simple substitution of a solution in form (64) into the boundary conditions. Here Φ belongs to class $C^2(I;I)$, the map is structurally stable. In particular, we can consider well-known unimodal maps [10], for example, the quadratic map $u\mapsto u^2+\mu$. For some $\mu\in R$, the maps have an infinite number of periodic points. Note that point u and trajectory $O(u)$ are called periodic with period m if $f^{(m)}(u)=u$, $f^{(j)}(u)\neq u$, $0<j<m$. For example, a periodic trajectory with period 2 contains two points u_0, $u_1=f(u_0)$, $f^{(2)}(u_0)=u_0$, $f^{(2)}(u_1)=u_1$. For $\mu=-2$ the map has an invariant measure which is absolutely continuous with respect to the Lebesgue measure. It means that the trajectories of the related dynamical system are «stochastic».

In a structurally stable case we define separator D of Φ as a set $D=\bigcup_{n\geq 0}h^{-n}\overline{P}^-$. Here $\overline{P}^-$ is closer of set P^- where P^- is a set of repelling points ofthe map. The separator represents a closed set of zero Lebesgue measure nowhere dense on interval I, which is finite, countable or uncountable. In particular, there is the following theorem [5]: D is uncountable if and only if Φ has circles with periods which are different from 2^i ($i=0,1,...$). Using D, we can construct a set $\Gamma = \tilde{h}^{-1}(D)$ where $\tilde{h}$ depends on the initial data of the boundary problem. In a structurally stable case, $h(t)$ satisfies the condition $h(t) \neq 0$, $t \in \Gamma$. Then topological properties of Γ are identical to the topological properties of separator D. Γ is closed and nowhere dense in [0; 1] with the measure $meas(\Gamma) = 0$. Γ determines a set of the points of 'discontinuities' for the solutions of the canonical system of equations in zero approximation (as $h = 0$).

The main statement of the present paper is that solutions of IBVP for the canonical system of equations are asymptotically stable in Skorohod or Hausdorff metrics if the small parameter $h < h_0$, where h_0 is determined by the parameters of the quantum problem. The Hausdorff metric is well-known. It is the distance between the graphics of solutions in the corresponding topology. This metric is applied to deterministic solutions. The Skorohod metric can be applied to the random solutions which represent an attractor of the problem. The Skorohod metric is [5]:

$$s(v,\tilde{v}) = \sup_{\alpha\in\Lambda}\left\{\|v\circ\alpha - \tilde{v}\|_{C^0(\Pi, I\times I)} + \|\alpha - Id\|_{C^0(\Pi,\Pi)}\right\} \tag{67}$$

Where Λ is a set of homeomorphisms, Id is identical homeomorphism. Further on, it will be shown that the solutions of the canonical problem are stable in zero approximation with respect to perturbations of the initial and boundary conditions in Skorohod and Hausdorff metrics. It must be noted that there exist a specific «stability» under specific initial conditions, which determine «solitons». Indeed, the initial functions must be taken from an area of attraction in zero approximation. Then it can be proved that in the succeeding approximations, all solutions of an attractive region tend to the limit solution in zero approximation as $t \mapsto \infty$ with accuracy $O(h^2)$, $O(h^3)$ for the first, second approximation, respectively, and so on. In this case, we deal with an approximated attractor of the original IBVP. We can confine ourselves by the approximation with accuracy $O(h)$.

The limit solution can be found, step by step, by the formula

$$p(t) = \Phi^{4m-1}\circ\Phi^{\Delta}\circ h\big(t - 2(2m-1)\big),\ \ t\in[4m-3, 4m-1),\ \ m = 1,2,... \tag{68}$$

where m is the least common multiple of the periods of attractive circles of the map $\Phi := \Phi_1 \circ \Phi_2$. A set of the points of 'discontinuities' is determined by the formula

$$\Upsilon_{R^+} = \bigcup_{n=1}^{\infty}\{t : t - 2n \in \Gamma\}. \tag{69}$$

7. The first approximation

Consider one of the components $\phi_1(x,t)$ of the system of transport equations. Initially, we selected the terms of order h^2 in the small parameter expansion of the original quantum equations. As a result, we obtain a system of uncoupled linear equations, which determine perturbations for zero approximation with accuracy $O(h^2)$:

$$\frac{\partial \phi_1^1}{\partial t}+\frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial \phi_1^1}{\partial x}=-\frac{i}{2}\frac{\partial^2 P_n^1}{\partial p^2}(\lambda_1)\frac{\partial^2 \phi_0^1}{\partial x^2}, \tag{70}$$

$$\frac{\partial \phi_2^1}{\partial t}+\frac{\partial P_n^2}{\partial p}(\lambda_2)\frac{\partial \phi_2^1}{\partial x}=-\frac{i}{2}\frac{\partial^2 P_n^2}{\partial p^2}(\lambda_2)\frac{\partial^2 \phi_0^2}{\partial x^2}. \tag{71}$$

Complex functions ϕ_1^1, ϕ_2^1 emerge because we use an «incorrect» expansion ([4], formula (19)):

$$\varphi(x,t)\equiv\sum_{j=0}^{\infty} h^j \varphi_j(x,t). \tag{72}$$

The correct expansion is

$$\varphi(x,t)\equiv\sum_{j=0}^{\infty} (ih)^j \varphi_j(x,t). \tag{73}$$

Indeed, it follows from the rigorous theory that general representation of solutions on the characteristics is ([6], p. 79):

$$u=u_0 \mathrm{e}^{i\tau S_0}+u_1 \mathrm{e}^{i\tau S_1}+\ldots+u_k \mathrm{e}^{i\tau S_k} \tag{74}$$

where $\tau=1/h$. The diffculty is that this expression does not coincide with the formula for the choice of phases S_j. If there is only one term in the sum, then this arbitrariness would result only in a rather harmless phase factor. However, in the case of several terms, the choice of the relative phases becomes essential. The correct oscillating terms in (74) are obtained from the projections of semi-density ρ that is a solution of the transport equation multiplied by a constant phase factor. These factors differ by degree *i*.

This problem can be studied by the example of the Lagrange manifolder Λ (see [6], p. 79). To solve of the problem, the method of stationary phase has been applied.

Indeed, the boundary conditions are

$$\phi_1^1=\Phi_1\left(\phi_2^1\right)\Big|_{x=0}, \quad \phi_2^1=\Phi_2\left(\phi_1^1\right)\Big|_{x=l}. \tag{75}$$

The main observation is that the right-hand parts of (70), (71) tend to zero as $t\to\infty$ for almost all characteristic of the difference equations. Then we may conclude that, as $t\to\infty$, the solutions of the boundary problem tend to solutions of the non-perturbed equations:

$$\frac{\partial \phi_1^1}{\partial t}+\frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial \phi_1^1}{\partial x}=0, \tag{76}$$

$$\frac{\partial \phi_2^1}{\partial t}+\frac{\partial P_n^2}{\partial p}(\lambda_2)\frac{\partial \phi_2^1}{\partial x}=0. \tag{77}$$

Then the problem is reduced to the Sharkovsky problem ([5], p. 247) (without the right-hand parts of the hyperbolic equations) with nonlinear boundary conditions:

$$\phi_1^0 + h\phi_1^1 = \Phi_1\left(\phi_2^0 + h\phi_2^1\right)\Big|_{x=0}, \quad \phi_2^0 + h\phi_2^1 = \Phi_2\left(\phi_1^0 + h\phi_1^1\right)\Big|_{x=1}. \tag{78}$$

Thus, it follows from (78) that

$$\phi_1^0 + h\phi_1^1 = \Phi_1\left(\phi_2^0\right) + h\,\Phi_1{}'\left(\phi_2^0\right)\phi_2^0\Big|_{x=0}, \; \phi_2^0 + h\phi_2^1 = \Phi_2\left(\phi_1^0\right) + h\,\Phi_2'\left(\phi_1^0\right)\phi_2^0\Big|_{x=l}, \tag{79}$$

8. Asymptotics for the quasi-invariant initial data

If $h = 0$, we obtain the well-known IBVP with typical attractors which represent a piecewise constant periodic function with finite or infinite lines of discontinuities that are located at characteristics of hyperbolic equations. Define, for simplicity, $\phi_1^0 = u_1$, $\phi_2^0 = u_2$. Then we find that

$$\Phi_1 \circ \Phi_2\left(u_1\right) = u_1, \quad u_2 = \Phi_2\left(u_1\right), \quad \Phi_1\left(u_2\right) = u_1. \tag{80}$$

Next we define $u_1\left(x,0\right) \equiv a_1$, where a_1 is a single attractive fixed point on interval I of the map $f := \Phi_1 \circ \Phi_2 : I \to I$. Put $u_2(x,0) = \Phi_2(u_1(x,0))$. Then the problem can be reduced to the difference equation [5]:

$$u_1\left(t\right) = f\left(u_1\left(t-\Delta\right)\right), \quad \Delta = l/V_1 + l/V_2, \tag{81}$$

where V_1, V_2 are coefficients of the hyperbolic equations, $V_1V_2 > 0$. Since $f \in C^2\left(I, I\right)$ has a single point $a_1 \in P^+$ where P^+ is a set of attractive fixed points, it follows from (81) that $u_1 \to a_1$, $u_2 \to \Phi_2(a_1)$ as $t \to \infty$. Further, it follows from structural stability of map f that the same statement is true if $\left(u_1\left(x,0\right), u_2\left(x,0\right)\right) \in \left(O_\delta\left(P^+, \Phi_1\left(P^+\right)\right)\right)$, where O_δ are some neighbourhoods of these points. Next, it is known [5] that if f is monotone (without an extremum) then set $P^+ = (a_1, ..., a_n)$ is finite. The values of piecewise constant limit function $p \in P^+$ almost at all points except a finite number of «jumps», where the value of p is a «vertical interval». In this case, we deal with the solutions of the relaxation type.

9. Asymptotics of the limit solutions

As a result, the solutions of the transport equations (41), (42) can be represented as

$$\phi_k\left(x,t\right) = \phi_0^1\left(t - \left(P_n^k\left(\lambda^k\right)\right)' x\right), \quad k = 1, 2. \tag{82}$$

Then it follows from (82) that the solutions of equations with perturbations can be represented as

$$\phi_k(x,t)=\phi_0^k\left(t-\left(P_n^k\left(\lambda^k\right)\right)'x\right)+ih\phi_1^k\left(t-\left(P_n^k\right)'\left(\lambda_k\right)x\right),\quad k=1,2. \tag{83}$$

The asymptotics of these solutions are

$$\phi_k(x,t)=\mathrm{e}^{i\pi/2}h\phi_1^k\left(t-\left(P_n^1\right)'\left(\lambda_k\right)x\right),\quad k=1,2. \tag{84}$$

10. The second approximation

In this case, with accuracy $O(h^3)$, we obtain a similar system of equations

$$\frac{\partial\phi_1^2}{\partial t}+\frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial\phi_1^2}{\partial x}=-\frac{i}{2!}\frac{\partial^2 P_n^1}{\partial p^2}(\lambda_1)\frac{\partial^2\phi_1^1}{\partial x^2}-\frac{i}{3!}\frac{\partial^3 P_n^1}{\partial p^3}(\lambda_1)\frac{\partial^3\phi_0^1}{\partial x^3}, \tag{85}$$

$$\frac{\partial\phi_2^2}{\partial t}+\frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial\phi_2^2}{\partial x}=-\frac{i}{2!}\frac{\partial^2 P_n^2}{\partial p^2}(\lambda_2)\frac{\partial^2\phi_2^2}{\partial x^2}-\frac{i}{3!}\frac{\partial^3 P_n^2}{\partial p^3}(\lambda_2)\frac{\partial^3\phi_0^2}{\partial x^3}. \tag{86}$$

Note that the second derivatives of the zero and first approximations tend to zero as time tends to infinity for almost all points on characteristics. It means that the limit asymptotics can be described by the limit equations:

$$\frac{\partial\phi_1^2}{\partial t}+\frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial\phi_1^2}{\partial x}=-\frac{i}{2!}\frac{\partial^2 P_n^1}{\partial p^2}(\lambda_1)\frac{\partial^2\phi_0^1}{\partial x^2}, \tag{87}$$

$$\frac{\partial\phi_2^2}{\partial t}+\frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial\phi_2^2}{\partial x}=-\frac{i}{2!}\frac{\partial^2 P_n^2}{\partial p^2}(\lambda_2)\frac{\partial^2\phi_0^2}{\partial x^2}. \tag{88}$$

Remind that for the succeeding approximations, the boundary conditions have the form

$$\phi_1^0+h\phi_1^1+h^2\phi_2^1=\Phi_1\left(\phi_2^0+h\phi_1^1+h^2\phi_2^1\right)\Big|_{x=0}, \tag{89}$$

$$\phi_2^0+h\phi_2^1+h^2\phi_2^2=\Phi_2\left(\phi_1^0+h\phi_1^1+h^2\phi_1^2\right)\Big|_{x=0}. \tag{90}$$

Then, as above, on the limit solution (p_1; p_2), where $p_1\in P^+$ and P^+ belongs to a set of attractive points of map $\Phi_1\circ\Phi_2$, $p_2=\Phi_2(p_1)$, we obtain linearised boundary conditions (89), (90)

$$\phi_1^1=\Phi_1'(p_1)\left(\phi_2^1\Big|_{x=0}\right.,\quad \phi_2^1=\Phi_2'(p_2)\left(\phi_1^1\Big|_{x=1}\right.. \tag{91}$$

Define $\phi_1^2\mapsto i\phi_1^2$ and $\phi_2^2\mapsto i\phi_2^2$. As a result, for nonperturbed system (85), (86), we obtain the difference equation

$$\phi_1^2(1,t+\Delta)=\Phi_1'(p)\Phi_2'(p)\left(\phi_1^2(1,t)\right),\quad \Delta=l/V_1+l/V_2. \tag{92}$$

Since $\left|\Phi_1'(p)\Phi_2'(p)\right| < 1$, then $\phi_1^2(1,t) \to 0$ as $t \to \infty$. For the non-perturbed system, function Φ_1^1 is of the same properties.

It will be shown below that for the perturbed system, the functions $\phi_1^k(x,t)$, $k = 1,2$ are of the same properties, too. Formally, it is possible because there is factor $\partial^2\phi_0^k(x,t)\big/\partial x^2$, $k = 1,2$, in the right-hand part of the perturbed system which tends to zero as $t \to \infty$.

Then, with accuracy $O(h^2)$, we obtain the following system

$$\frac{\partial u_1}{\partial t} + \frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial u_1}{\partial x} = F_1\left(\phi_0^1\right), \tag{93}$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial P_n^2}{\partial p}(\lambda_2)\frac{\partial u_2}{\partial x} = F_2\left(\phi_0^1\right), \tag{94}$$

where $u_1 = \phi_1^1$, $u_2 = \phi_2^1$.

The boundary conditions are

$$u_1 = \Phi_1'(p)u_2\big|_{x=0}, \quad u_2 = \Phi_2'(p)u_1\big|_{x=1}. \tag{95}$$

The problem of existence and uniqueness of solutions has been considered in [11]. Next, by integration along the characteristics, we can show that this solution satisfies a system of integro-difference equations:

$$u_1(l,t_0) = u_1(0,t_0 - l/V_1) + V_1\int_{t_0-l/V_1}^{t_0}\frac{\partial^2\phi_0{}^1}{\partial x^2}\left(V_1 t - V_1(t_0 - l/V_1),t\right)\mathrm{d}t =$$

$$= \Phi_1\left(u_2(0,t_0 - l/V_1)\right) + V_1\int_{t_0-l/V_1}^{t_0}\frac{\partial^2\phi_0{}^1}{\partial x^2}\left(V_1 t - V_1(t_0 - l/V_1),t\right)\mathrm{d}t =$$

$$= \Phi_1\left(\Phi_2\left(u_1(0,t_0 - l/V_1 - l/V_2)\right) + V_2\int_{t_0-l/V_1-l/V_2}^{t_0-l/V_1}\frac{\partial^2\phi_0{}^2}{\partial x^2}\left(V_2 t - V_2(t_0 - l/V_1),t\right)\mathrm{d}t\right) +$$

$$+ V_1\int_{t_0-l/V_1}^{t_0}\frac{\partial^2\phi_0{}^1}{\partial x^2}\left(V_1 t - V_1(t_0 - l/V_1),t\right)\mathrm{d}t, \tag{96}$$

$$u_2(l,t_0) = u_2(l,t_0 - l/V_2) + V_2\int_{t_0-l/V_2}^{t_0}\frac{\partial^2\phi_0{}^2}{\partial x^2}\left(V_2 t + l/V_2 - t_0,t\right)\mathrm{d}t =$$

$$= \Phi_2\left(\Phi_1\left(u_2(l,t_0 - l/V_1 - l/V_2)\right) + V_1\int_{t_0-l/V_1-l/V_2}^{t_0-l/V_2}\frac{\partial^2\phi_0{}^1}{\partial x^2}\left(V_1(t - t_0 + l/V_2 + l/V_1),t\right)\mathrm{d}t\right) +$$

$$+ V_2 \int\limits_{t_0 - l/V_2}^{t_0} \frac{\partial^2 \phi_0{}^2}{\partial x^2}\left(V_2 t + l/V_2 - t_0, t\right) \mathrm{d}t \,. \tag{97}$$

We deduce from (96), (97) that

$$u_1(l,t_0) = u_1(0,t_0 - l/V_1) + l\frac{\partial^2\phi_0^1}{\partial x^2}(t_0 - l/V_1) = \Phi_1\left(u_2(0,t_0 - l/V_1)\right) + l\frac{\partial^2\phi_0^1}{\partial x^2}(t_0 - l/V_2) =$$

$$= \Phi_1\left(\Phi_2\left(u_1(0,t_0 - l/V_1 - l/V_2)\right) + l\frac{\partial^2\phi_0^1}{\partial x^2}(t_0 - l/V_1)\right) + l\frac{\partial^2\phi_0^2}{\partial x^2}(t_0 - l/V_2), \tag{98}$$

$$u_2(l,t_0) = u_2(l,t_0 - l/V_2) + l\frac{\partial^2\phi_0^1}{\partial x^2}(t_0 - l/V_1) = \Phi_1\left(u_2(0,t_0 - l/V_1)\right) + l\frac{\partial^2\phi_0^1}{\partial x^2}(t_0 - l/V_2) =$$

$$= \Phi_2\left(\Phi_1\left(u_2(l,t_0 - l/V_1 - l/V_2)\right) + l\frac{\partial^2\phi_0^1}{\partial x^2}(t_0 - l/V_1)\right) + l\frac{\partial^2\phi_0^1}{\partial x^2}(t_0 - l/V_2). \tag{99}$$

Note that one of components $\phi_0^{1,2}$ satisfies the difference equation

$$u(\zeta) = G\left(u(\zeta - \Delta)\right) \tag{100}$$

where $\Delta = l/V_1 + l/V_2$ and $G := \Phi_1\Phi_2$ or $G := \Phi_2\Phi_1$. Since G is hyperbolic, it follows from (100) that

$$u'(\zeta) = G'\left(u(\zeta - \Delta)\right)u'(\zeta - \Delta) \tag{101}$$

where $u'(\zeta) \in O_\gamma\left(P^+\right)$ and P^+ is a set of attractive points of the map. Then

$$u''(\zeta) = G\left(u(\zeta - \Delta)\right)\left(u'(\zeta - \Delta)\right)^2 + G'\left(u(\zeta - \Delta)\right)u''(\zeta - \Delta). \tag{102}$$

It follows from (102) that

$$\left|u'(\zeta)\right| \le \lambda \left|u'(\zeta - \Delta)\right|, \tag{103}$$

where $\lambda < 1$. Hence, $\left|u'(\zeta)\right| \to 0$ as $t \to +\infty$. Then we obtain from (102) that $\left|u''(\zeta)\right| \to 0$ as $t \to +\infty$. The linearised equation

$$u(\zeta) = \lambda u(\zeta - \Delta) \tag{104}$$

has a positive solution $u(\zeta) = u(\zeta_0)\mathrm{e}^{k(t - t_0)}$ where $k = \frac{1}{\Delta}\ln\lambda$ and $\lambda < 1$ at each fixed point. Thus

$$\frac{\partial^2\phi_0^j}{\partial x^2}\left(t - l/V_j\right) = \left(\frac{k_j}{V_j}\right)^2 \mathrm{e}^{k_j\left(t_0 - l/V_j\right)}, \quad j = 1,2, \quad k_j < 0\,. \tag{105}$$

From (98), (99) (105), we arrive at

$$u_1(l,t_0) = u_1(0,t_0 - l/V_1) + l\left(\frac{k_1}{V_1}\right)^2 \mathrm{e}^{k_1(t_0 - l/V_1)} = \Phi_1\left(u_2(0,t_0 - l/V_1)\right) + l\left(\frac{k_2}{V_2}\right)^2 \mathrm{e}^{k_2(t_0 - l/V_2)} =$$

$$= \Phi_1\left(\Phi_2\left(u_1(0,t_0 - l/V_1 - l/V_2)\right) + l\left(\frac{k_1}{V_1}\right)^2 \mathrm{e}^{k_1(t_0 - l/V_1)}\right) + l\left(\frac{k_2}{V_2}\right)^2 \mathrm{e}^{k_2(t_0 - l/V_2)}, \quad (106)$$

$$u_2(l,t_0) = u_2(l,t_0 - l/V_2) + l\left(\frac{k_1}{V_1}\right)^2 \mathrm{e}^{k_1(t_0 - l/V_1)} =$$

$$= \Phi_2\left(\Phi_1\left(u_2(l,t_0 - l/V_1 - l/V_2)\right) + l\left(\frac{k_1}{V_1}\right)^2 \mathrm{e}^{k_1(t_0 - l/V_1)}\right) + l\left(\frac{k_1}{V_1}\right)^2 \mathrm{e}^{k_1(t_0 - l/V_1)}. \quad (107)$$

Without a loss of generality, we assume that $\Phi_2 := Id$ where Id is an identical map. Then we obtaint from the above equations that

$$u_1(l,t_0) = \Phi_1\left(u_1(0,t_0 - l/V_1 - l/V_2) + \left(\frac{k_2}{V_1}\right)^2 \mathrm{e}^{k_1(t_0 - l/V_1)}\right) + l\left(\frac{k_2}{V_2}\right)^2 \mathrm{e}^{k_1(t_0 - l/V_2)}, \quad (108)$$

$$u_2(l,t_0) = \Phi_1\left(u_2(0,t_0 - l/V_1 - l/V_2) + \left(\frac{k_1}{V_1}\right)^2 e^{k_1(t_0 - l/V_1)}\right) + l\left(\frac{k_1}{V_1}\right)^2 e^{k_2(t_0 - l/V_1)}. \quad (109)$$

Since $k_{1,2}$ are negative it is easy to see that in these difference equations with non-autonomic perturbations, the exponential factors tend to zero as $t \to +\infty$. Then it can be shown that an asymptotics of solutions can be determined as asymptotics of limit difference equations

$$u_1(l,t_0) = \Phi_1\left(u_1(l,t_0 - \Delta)\right) + l\left(\frac{k_2}{V_2}\right)^2 \mathrm{e}^{k_1(t_0 - l/V_2)}, \quad (110)$$

$$u_2(l,t_0) = \Phi_1\left(u_2(l,t_0 - \Delta)\right), \quad \Delta = l/V_1 - l/V_2. \quad (111)$$

But we know that for an unimodal map (with one extremum), the solutions of this equations tend to piecewise constant periodic functions with finite or infinite points of discontinuities on a period.

11. Applications to non-coherent optical solitons

The phenomenon of appearing optical solutions is determined by the dynamical balance between the concurrence of two factors: (1) detention of the optical beam to expand over the media provided by diffraction; (2) detention of the beam to restrict the media due to self-focusing [12]. Experiments (see [12]) show the possibility of the existence of solitons which are spatially non-coherent and quasi-monochromatic or non-coherent together with spatial-temporal variables. These experiments initiated a set of theoretical works which concern to the non-coherent solitons (see [12,13]). However, these works have been confined by the research of the last case. It means that the corresponding theory could not model non-coherent white light for example that is studying of spatial-temporal coherent properties of the solitons and the evolution of the spectral density. In the present paper this problem is studied, and in this section a simplest clear example of such situation will be considered. Indeed, below we consider the light that is spatial-temporal on (*x*; *t*). We assume that the spatial profile of the light belongs to the interval of frequencies $[\omega, \omega + \mathrm{d}\omega]$; spatial correlation length (across of a soliton) is always larger at low frequencies and smaller at high frequencies (see [12]).

We begin the research from the following equation:

$$i\left(\frac{\partial f^{\omega}}{\partial z} + \theta\frac{\partial f^{\omega}}{\partial x}\right) + \frac{1}{2k_{\omega}}\frac{\partial^2 f^{\omega}}{\partial x^2} + \frac{k_{\omega}}{n_0}\delta n(I) f^{\omega}(x,z,\theta) = 0. \quad (112)$$

Here f^{ω} is the coherent density of the optical beam at a fixed frequency, $k_{\omega} = n_0\,\omega/c$ where n_0 is the refractive index, ω is frequency, *c* is the velocity of light, θ determines the angle between the direction of light (at plane (*z*; *x*)) and *Oz* axes.

Spatial-temporal coherent properties of a beam may be studied in terms of spectral density

$$B_{\omega}(x_1,x_2,z) = \int_{-\infty}^{+\infty} \mathrm{d}\theta\, \exp\left[ik_{\omega}(x_1 - x_2)\right] f^{\omega}(x_1,z,\theta) f^{\omega}(x_2,z,\theta). \quad (113)$$

Note that equation (112) is equivalent to the related equation (113).

We suppose that an optical medium is dispersive. If we assume that $\partial\delta n(I)/\partial t \equiv 0$ then the dispersion may be included in the consideration with the aid of dependence $n_0 = n_0(\omega)$. Then instead of the classical equation (112), we consider the Shrödinger equation in laboratory system of coordinate with an optical source confined by the one-dimensional case.

Note that semiconductor lasers or laser diodes were considered in [2]. The laser is a system characterized by the inverted carrier density. The generation and recombination of «solitons» coexist. The released energy can be produced by thermal recombination or optical photon recombination, which is used in semiconductor lasers. Note also that the electronic oscillator is an electronic circuit that produces a periodic signal. Oscillators convert direct current to an alternative current signal. If we use the feedback oscillator, which can increase amplitudes of signal, then we obtain different boundary conditions for phases and amplitudes in the canonical equations.

For example, let us consider the region $0 < x < l.\ z \geq 0$, which is occupied by a resonator. The equations have the form

$$ih\left(\frac{\partial f}{\partial t}+\theta\frac{\partial f}{\partial x}\right)+\frac{h^2}{2k}\frac{\partial^2 f}{\partial x^2}+\frac{k}{n_0}\delta n(I)f=0 \quad (114)$$

where index ω is omitted.

The solutions of (114) are found as $f:=\varphi(\zeta,t)$ where $\zeta=t-x/V$. Then it follows from (114) that

$$-ih\frac{\partial\varphi}{\partial t}+\frac{h^2}{2k}\frac{\partial^2\varphi}{\partial\zeta^2}-\frac{k}{n_0}\delta n(I)f=0. \quad (115)$$

Let $\bar{x}=\sqrt{k}x$. Then this equation can be written as

$$-ih\frac{\partial\varphi}{\partial t}+\frac{h^2}{2}\frac{\partial^2\varphi}{\partial\zeta^2}-\frac{k}{n_0}\delta n(I)f=0. \quad (116)$$

The Shrödinger equation has the form

$$-ih\frac{\partial\varphi}{\partial t}+\frac{h^2}{2}\frac{\partial^2\varphi}{\partial\zeta^2}=0 \quad (117)$$

with the special initial conditions

$$\varphi(x,0)=\varphi_0 e^{iS_0(x)/h} \quad (118)$$

where V, S_0, φ_0 are smooth real functions. The Hamiltonian is

$$H(p,q)=\frac{p^2}{2}+V(q). \quad (119)$$

Asymptotic solutions of the initial problem (117), (118) have the form

$$\varphi(x,t)=e^{iS_0(x,t)/h}\varphi(x,t) \quad (120)$$

where unknown functions $S(x,t)$ and $\varphi(x,t)$ are smooth. Substituting (120) into (117), we obtain the equation:

$$\left[S_t+V(x)+\frac{1}{2}(S_x)^2\right]\varphi+(-ih)\left[S_x\varphi_x+\varphi_t+\frac{1}{2}\varphi S_{xx}\right]+\left(-\frac{ih}{2}\right)\varphi_{xx}=0. \quad (121)$$

It follows from (121) that

$$S_t+V(x(\zeta,t))+\frac{1}{2}S_\zeta^2=0,\quad S(\zeta,0)=S_0(\zeta) \quad (122)$$

with the accuracy of $O(h^2)$ where $\varphi(\zeta,t)$ satisfies the initial problem

$$\varphi_e+\varphi_\zeta S_\zeta+\frac{1}{2}\varphi S_{\zeta\zeta}=0,\quad \varphi(\zeta,0)=\varphi_0(\zeta). \quad (123)$$

The result is

$$S_t + \frac{1}{2} S_\zeta^2 = 0 . \quad (124)$$

The solution of (124) has the form $S(\zeta, t) = \lambda_1 t + \lambda_2 \zeta$ that results in an algebraic relation

$$\lambda_1 + \lambda_2^2 = 0 . \quad (125)$$

It follows from (125) that

$$S(\zeta, t) = \lambda_2 \left(\zeta - \frac{1}{2} \lambda_2 t \right) \quad (126)$$

where λ_2 can be devived from

$$S(\zeta_{t=0}, 0) = \lambda_2 \zeta_{t=0} = -\frac{\lambda_2}{\theta} x = S_0(x) . \quad (127)$$

Thus, the phase has the form as follows:

$$S(x, t) = \lambda_1 t + \lambda_2 \left(t - \frac{x}{\theta} \right). \quad (128)$$

Since the phase is linear, we have $S_{\zeta\zeta} = 0$ and $S_\zeta = 0$, so the equation is rewritten as

$$\varphi(\zeta, t) := \varphi\left(t - \frac{\zeta}{\lambda_2} \right). \quad (129)$$

Now we consider the boundary conditions

$$\varphi_t(0, t) = F_1[\varphi(0, t)], \quad \varphi_t(l, t) = F_2[\varphi(l, t)], \quad t > 0 \quad (130)$$

where F_1 and F_2 are fixed functions.

Assume that the system of ODEs is integrable, that is there exists the integral

$$W[\varphi(0, t), \varphi(l, t)] = \mu , \quad \mu \in R . \quad (131)$$

Suppose that there is an open bounded interval $I \subset R^+$ such that for all $\varphi(0, t)$, $\varphi(l, t) \in I$ and at each fixed $t > 0$ relation (131) is solvable so that

$$\varphi(l, t) = \Phi_\mu[\varphi[0, t]] \quad (132)$$

where $\Phi_\mu : I \mapsto I$ is a unimodal map of C^2-class. Then

$$\varphi(l, t) = \varphi\left[\left(1 - \frac{1}{\lambda_2} \right) t + \frac{l}{\lambda_2 \theta} \right], \quad (133)$$

$$\varphi(0, t) = \varphi\left[\left(1 - \frac{1}{\lambda_2} \right) t \right]. \quad (134)$$

Let $\lambda = 1 - \lambda_2^{-1}$, $t \mapsto \lambda t$ and $L = l/\lambda_2\theta$. Then functional quality (133) can be written as

$$\varphi(t+l) = \Phi_\mu \varphi(t), \quad -l < t < +\infty. \tag{135}$$

As a result, we obtain a difference equation [3,14]. If Φ_μ is unimodal, structurally stable and hyperbolic then a set of fxed points of this map is finite. Then there is a set of initial functions $h(t)$, $t \in [-L, 0)$ such that the solutions of the difference equation can be found by step-by-step iterations of the initial function $h(t)$ with the help of Φ_μ. As a result, for $t \to \infty$, the iterations of $h(t)$ tend to a periodic piecewise constant function with finite or infinite points of discontinuities Γ on a period. If Γ is finite then we say about oscillations of the relaxation type. If Γ is countable then we have oscillations of the pre-turbulent type. If Γ is uncountable then we have oscillations of the turbulent type.

1. *C. Beck*, Physica **D125**, 171 (1999).
2. *I.B. Krasnyuk*, Journal of Applied Mathematics and Physics **9**, 1714 (2016).
3. *A.N. Sharkovsky, E.Yu. Romanenko*, Int. J. Bifurcat. Chaos **2**, № 1, 31 (1992).
4. *A.S. Mishchenko, B.Yu. Sternin, V.E. Shatalov*, Lagrangian manifolds and the Maslov operator, Springer, Berlin (1990).
5. *A.N. Sharkovsky, Yu.L. Maistrenko, E.Yu. Romanenko*, Difference Equations and Their Applications. Ser. Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht (1993).
6. *V. Guillemin, S. Sternberg*, Geometric Asymptotics, Mathematical Surveys and Monographs, Vol. 14 (1977).
7. *V.P. Maslov*, Complex WKB method in nonlinear equations, Nauka, Moskow (1977) (in Russian).
8. *A.M. Gulian, G.F. Gharkov*, Superconductors in permanent fields, Nauka, Moskow (1990) (in Russian).
9. *I. Krasnyuk*, Quantum Physics Letters **6**, № 1, 13 (2017).
10. *A.N. Sharkovsky, S.F. Kolyada, A.G. Sivak, V.V. Fedorenko*, Dynamics of one-dimensional maps, Naukova Dumka, Kiev (1989).
11. *V.E. Abolinya, A.D. Myshkis*, A mixed problem for an almost linear hyperbolic system on the plane, Mathem. Sb. **50(92)**, 423 (1960) (in Russian).
12. *H. Buljan, M. Segev, M. Soljacic, N.K. Efremidis, D.N. Christodoulides*, Opt. Lett. **28**, 1239 (2003).
13. *A.I. Maimistrov*, Sorovsky general educational journal **11**, 97 (1999).
14. *E.Yu. Romanenko*, Autostochasticity of difference equations with continuous time, Pre-print 2008.2, Institute of Mathematics NANU, Kiev (2008).

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ВОЛНОВЫЕ ПАКЕТЫ ТУРБУЛЕНТНОГО ТИПА В НЕЛИНЕЙНЫХ ГРАНИЧНЫХ ЗАДАЧАХ КВАНТОВОЙ МЕХАНИКИ

Рассмотрена начально-краевая задача для линейного уравнения Шредингера с нелинейными функциональными граничными условиями. Показано, что аттрактор задачи содержит периодические кусочно-постоянные функции на комплексной плоскости с конечным числом точек разрыва непрерывности на периоде. Предложен метод сведения задачи к системе интегро-дифференциальных уравнений. Рассмотрено приложение к задаче об оптическом резонаторе с обратной связью. Элементы аттрактора могут трактоваться как белые и черные солитоны в нелинейной оптике.

Ключевые слова: уравнение Шредингера, функциональные двухточечные граничные условия, асимптотические периодические кусочно-постоянные распределения релаксационного типа

Рис. 1. Траектории гиперболических динамических систем с притягивающими и седловыми точками на плоскости

Рис. 2. Ограниченные решения релаксационного типа