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FINITE SPEED OF THE ELECTROMAGNETIC-FIELD PROPAGATIONS IN NONLINEAR ISOTROPIC DISPERSIVE MEDIA

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A modification of Maxwell's equations is proposed to describe media with electric and magnetic properties changing essentially under electromagnetic field. It is shown that for such media the electromagnetic waves have finite speed of propagations property for a time depending on the initial energy of electromagnetic field and nonlinear parameters of the medium.

Keywords: Maxwell's equations, nonlinear dispersive medium, finite speed of propagations, asymptotic behavior

1. Introduction

We consider a classical Maxwell system in the SI (see [12]):

$$(M_0) \quad \begin{cases} \mathbf{D}_t + \mathbf{J} = \operatorname{curl}\mathbf{H}, & (1.1) \\ \mathbf{B}_t + \operatorname{curl}\mathbf{E} = 0, & (1.2) \\ \operatorname{div}\mathbf{D} = \rho, \operatorname{div}\mathbf{B} = 0, & (1.3) \end{cases}$$

where $\mathbf{E}$ and $\mathbf{H}$ are electric and magnetic fields; $\mathbf{D}$ and $\mathbf{B}$ are electric and magnetic inductions; ρ is the charge density. The current density $\mathbf{J}$ satisfies the Ohm's law:

$$\mathbf{J} = \sigma\mathbf{E}, \quad (1.4)$$

where σ is the electric conductivity.

We consider isotropic media in which permittivity $\varepsilon = \varepsilon(x,t)$ and magnetic $\mu = \mu(x,t)$ conductivity are functions of space and time. In this situation, equations of state have the following simple form (see [12]):

$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}. \quad (1.5)$$

Substituting (1.5) into equations (1.1) and (1.2) we obtain equations for $\tilde{\mathbf{E}} = \mathbf{E}/E_0$ and $\tilde{\mathbf{H}} = \mathbf{H}/H_0$ for an isotropic medium in the following dimensionless form:

$$(\tilde{M}_0) \quad \begin{cases} \tilde{\mathbf{E}}_{\tilde{t}} + \tilde{a}_1\tilde{\mathbf{E}} - \tilde{b}_1\text{curl}\tilde{\mathbf{H}} = 0, & (1.6) \\ \tilde{\mathbf{H}}_{\tilde{t}} + \tilde{a}_2\tilde{\mathbf{H}} + \tilde{b}_2\text{curl}\tilde{\mathbf{E}} = 0, & (1.7) \end{cases}$$

where $\tilde{t} = \dfrac{t}{t_0}$, $\tilde{x}_i = \dfrac{x_i}{x_0}$, $\tilde{\varepsilon} = \dfrac{\varepsilon}{\varepsilon_0}$, $\tilde{\mu} = \dfrac{\mu}{\mu_0}$, $\tilde{\sigma} = \dfrac{\sigma}{\sigma_0}$, and

$$\tilde{a}_1 = \frac{1}{\tilde{\varepsilon}}\left(\frac{d\tilde{\varepsilon}}{dt} + \frac{\sigma_0 t_0}{\varepsilon_0}\tilde{\sigma}\right), \quad a_2 = \frac{1}{\tilde{\mu}}\frac{d\tilde{\mu}}{dt}, \quad b_1 = \frac{t_0 H_0}{x_0\varepsilon_0 E_0}\frac{1}{\tilde{\varepsilon}}, \quad b_2 = \frac{t_0 E_0}{x_0\mu_0 H_0}\frac{1}{\tilde{\mu}}. \quad (1.8)$$

Here the subscript 0 denotes the corresponding values for vacuum. Further, we suppose that

$$\frac{\sigma_0 t_0}{\varepsilon_0} = 1, \quad \frac{t_0 H_0}{x_0\varepsilon_0 E_0} = 1, \quad \frac{t_0 E_0}{x_0\mu_0 H_0} = 1,$$

whence we find that

$$\frac{H_0}{E_0} = c\varepsilon_0, \quad t_0 = \frac{\varepsilon_0}{\sigma_0}, \quad x_0 = \frac{c\varepsilon_0}{\sigma_0}, \quad (1.9)$$

where $c = \dfrac{1}{\sqrt{\mu_0\varepsilon_0}}$ is the velocity of light. In view of (1.9), omitting '⊔', we obtain the following system:

$$(M_1) \quad \begin{cases} \mathbf{E}_t + a_1\mathbf{E} - b_1\text{curl}\mathbf{H} = 0, & (1.10) \\ \mathbf{H}_t + a_2\mathbf{H} + b_2\text{curl}\mathbf{E} = 0, & (1.11) \end{cases}$$

where

$$a_1 = \frac{1}{\varepsilon}\left(\frac{d\varepsilon}{dt} + \sigma\right), \quad a_2 = \frac{1}{\mu}\frac{d\mu}{dt}, \quad b_1 = \frac{1}{\varepsilon}, \quad b_2 = \frac{1}{\mu}. \quad (1.12)$$

Hyperbolic systems as (M_1) are well investigated (see, e.g., [21]). For example, if $\sigma = 0$, and

$$\mathbf{D} = \varepsilon(\mathbf{E}, \mathbf{H})\mathbf{E}, \quad \mathbf{B} = \mu(\mathbf{E}, \mathbf{H})\mathbf{H}, \quad (1.13)$$

then the system (M_0), (1.13) is that considered in [6]. In particular, the authors have shown that this system is Poincaré-invariant if and only if $\varepsilon(\mathbf{E}, \mathbf{H})\mu(\mathbf{E}, \mathbf{H}) = c^2$, and they found its various invariants.

Note, the system (M_1) can be reduced to the following equation relative to $\mathbf{E}$:

$$\mathbf{E}_{tt} + a_3\mathbf{E}_t + a_4\mathbf{E} + b_1(b_2\nabla\,\text{div}\,\mathbf{E} - b_2\Delta\mathbf{E} + \nabla b_2 \times \text{curl}\,\mathbf{E} + \nabla a_2 \times \mathbf{H}) = 0, \quad (1.14)$$

where

$$a_3 = \frac{1}{\varepsilon}\left(\frac{d\varepsilon}{dt} + \sigma\right) + \frac{1}{\varepsilon}\frac{d\varepsilon}{dt} + \frac{1}{\mu}\frac{d\mu}{dt}, \quad a_4 = \frac{d}{dt}\left(\frac{1}{\varepsilon}\left(\frac{d\varepsilon}{dt} + \sigma\right)\right) + \frac{1}{\varepsilon}\left(\frac{d\varepsilon}{dt} + \sigma\right)\left(\frac{1}{\varepsilon}\frac{d\varepsilon}{dt} + \frac{1}{\mu}\frac{d\mu}{dt}\right).$$

Now, we consider the special case when $\varepsilon = \varepsilon(t)$, $\mu = \mu(t)$ such that $\varepsilon(t)\mu(t) = v^2$ is a constant. In this case, due to (1.13), the equation (1.14) has the form:

$$\mathbf{E}_{tt} + (a_5\mathbf{E})_t - v^{-2}\Delta\mathbf{E} = 0, \tag{1.15}$$

where $a_5 = \dfrac{1}{\varepsilon}\left(\dfrac{d\varepsilon}{dt} + \sigma\right)$. For example, if a_5 is a constant, i.e. $\varepsilon(t) =$

$$= e^{a_5 t}\left(\varepsilon(0) - \int_0^t \sigma(\tau)e^{-a_5\tau}d\tau\right),$$ then (1.15) has the following solution $\mathbf{E}_w =$

$$= \left\{E_{w,j}\right\} = \left\{e^{ik_j x + \omega_j t}\right\}_{j=1}^N,$$ where $k_j \in R^N$, and $\omega_j = \dfrac{-a_5 \pm \sqrt{a_5 - 4v^{-2}\left|k_j\right|^2}}{2}$. As a

consequence, this solution has spatiotemporal oscillations if $a_5 - 4v^{-2}\,|\,k_j\,|^2 < 0$, and spatial oscillations only if $a_5 - 4v^{-2}\,|\,k_j\,|^2 \geq 0$.

In a more general situation, electric and magnetic inductions depend on electric and magnetic fields (see [12]), i.e.

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H}), \ \ \mathbf{B} = \mathbf{B}(\mathbf{E}, \mathbf{H}). \tag{1.16}$$

Below, we consider the simplest case of the equations of state (1.16) when permittivity and magnetic conductivity are some functions of space and time depending on electric E and magnetic H fields and its gradients, precisely, of the energy density w and ∇w. In the case, we arrive at the system (M_1), i.e. equations for $\mathbf{E}$ and $\mathbf{H}$ in an isotropic nonlinear medium, where $a_i = a_i(x,t,w,\nabla w)$ and $b_i = b_i(x,t,w,\nabla w)$ ($i = 1, 2$) satisfy relations (1.12). We will study media in which nonlinear functions a_i and b_i satisfy the following conditions:

$$a_i(x,t,w,\nabla w) \geq d_1 w^{m-1}\,|\nabla w|^p, \ \ 0 < d_1 < \infty, \ i = 1, 2, \tag{1.17}$$

$$b_1(x,t,w,\nabla w) = b_2(x,t,w,\nabla w), \tag{1.18}$$

$$|\,b_i(x,t,w,\nabla w)\,| \leq d_2 w^{n-1}, \ \ 0 < d_2 < \infty, \ i = 1, 2, \tag{1.19}$$

$$|\,\nabla b_i(x,t,w,\nabla w)\,| \leq d_3 w^{n-2}\,|\nabla w|, \ \ 0 < d_3 < \infty, \ i = 1, 2, \tag{1.20}$$

where $w = w(x,t) = E^2 + H^2$ is the dimensionless energy density corresponding to isotropic media with constant permittivity and magnetic conductivity;

$$m \in R^1, \ p > 0 \text{ and } n > 0 \tag{1.21}$$

are parameters of medium. The special choice of structure conditions on a_i and b_i allows us to apply methods from the theory of parabolic equations to the hyperbolic system (M_1). Conditions (1.17)–(1.20) result in the following restrictions on ε and μ:

$$\varepsilon = \mu \ge d_2^{-1} w^{1-n}, \quad \frac{1}{\varepsilon}\left(\frac{d\varepsilon}{dt} + \sigma\right) \ge d_1 w^{m-1} \, |\nabla w|^p \, , \quad \frac{1}{\mu}\frac{d\mu}{dt} \ge d_1 w^{m-1} \, |\nabla w|^p \, ,$$

whence we deduce that

$$\varepsilon = \mu \ge \max\left\{ d_2^{-1} w^{1-n}, \;\; \varepsilon\big|_{t=0} \exp\left(d_1 \int_0^t w^{m-1} \, |\nabla w|^p \, d\tau \right) \right\}.$$

The equations like (M_1) describe media in which permittivity and magnetic conductivity are some nonlinear functions. The media have same structure to have to appear in the simulation of various processes in laser optics and weakly ionized plasma theory, where properties of medium are strongly dependent on energy density of electromagnetic field, for example, ferroelectric, piezoelectric, multiferroic, etc.

In this paper, we study the propagation properties of solutions to Cauchy problem for Maxwell's equations in the following dimensionless form

$$(M) \quad \begin{cases} \mathbf{E}_t + a_1 \, \mathbf{E} - b_1 \, \mathrm{curl} \, \mathbf{H} = 0 \; \text{in} \; Q_T, & (1.22) \\ \mathbf{H}_t + a_2 \, \mathbf{H} + b_2 \, \mathrm{curl} \, \mathbf{E} = 0 \; \text{in} \; Q_T, & (1.23) \\ \mathbf{E}(0,x) = \mathbf{E}_0(x), \;\; \mathbf{H}(0,x) = \mathbf{H}_0(x), & (1.24) \end{cases}$$

where $Q_T = (0,T) \times R^N$, $N = 2, 3$, $0 < T < \infty$, and the functions $a_i = a_i(x,t,E,H,\nabla E,\nabla H)$, $b_i = b_i(x,t,E,H,\nabla E,\nabla H)$ ($i = 1, 2$) satisfy conditions (1.17)–(1.20). The unknown functions are electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields, which depend on the time t and the space-variable x. Moreover, we suppose that the initial electromagnetic field is located in half-space $R_-^N := \left\{ x = (x', x_N) \in R^N : x_N < 0 \right\}$, i.e.

$$\mathrm{supp} \, w(.,0) \subset R_-^N \, , \qquad (1.25)$$

where $w(x,t) = E^2 + H^2$.

Remark 1.1. If $a_1 = a_2$, $b_1 = b_2$, and $\mathrm{div}(\mathbf{E} \times \mathbf{H}) = 0$ then from (M) we find that $w(x,t) = w(x,0)\exp\left(-2\int_0^t a_1 \, d\tau\right)$, i.e. the energy density w decays in time.

Thus, the presented system (M) is obtained from the classical Maxwell's system (M_0) taking into account the equations of state (1.16) for isotropic nonlinear medium and the Ohm's law for current density (1.4). Media are described to possess the finite speed propagations property. There are many papers in which energy decay was obtained for different problems concerning Maxwell's equations. Well-posedness and asymptotic stability results and decay of solutions are proved making use of different techniques. Below, we mention some results concerning energy decay and asymptotic of solutions.

Some linear evolution problems arise in the theory of hereditary electromagnetism. Many authors studied the influence of dissipation due to the memory on

the asymptotic behavior of the solutions (see [2,4,5,7,13,14,20]). The polynomial decay of the solutions when the memory kernel decays exponentially or polynomially was shown in [15]. It is studied the asymptotic behavior of the solution of the linear problem describing the evolution of the electromagnetic field inside a rigid conducting material, whose constitutive equations contain memory terms expressed by convolution integrals. These models were proposed in [19] where it was shown that the exponential decay of the memory kernel is able to produce a uniform rate decay of the energy in rigid conductors with electric memory.

The exact boundary controllability and stabilization of Maxwell's equations have been studied by many authors (see [17] and references therein). In [17] the internal stabilization of Maxwell's equations with the Ohm's law for space variable coefficients is studied. The authors give sufficient conditions on parameters of the medium which guarantee the exponential decay of the energy of the system. The result is based on observability estimate, obtained in some particular cases by the multiplier method, a duality argument and a weakening of norm argument, and argument used in internal stabilization of scalar wave equations.

The energy decay of solutions of the scalar wave equation with nonlinear damping in bounded domains has been shown in [3,11,16,23–26]. In the case when there is no damping term in the equation for the dielectric polarization, the long-time asymptotic behavior of the solution of Maxwell's equations involving generally nonlinear polarization and conductivity is studied in [8].

The propagation of electromagnetic waves in gas of quantum mechanical system with two energy levels is considered in [10]. The decay of the polarization field in a Maxwell–Bloch system for $t \to \infty$ was shown.

The transient Landau–Lifschitz equations describing ferromagnetic media without exchange interaction coupled with Maxwell's equations is considered in [9]. The asymptotic behavior of the solution of this mathematical model for micromagnetism is studied. It is shown the strong convergence of the electromagnetic field with respect to the energy norm for $t \to \infty$ on bounded sets of nonvanishing electrical conductivity.

Following the dominant trend in the literature, we can conclude that study of the system (*M*) is not only of theoretical interest but is useful for applied researches. Since these authors are not specialists in electromagnetism, we apologize in advance for the omissions and inaccuracies. We hope that there is an interdisciplinary audience which may find this useful, whether we do not know any concrete media with the proposed properties.

The present paper is organized as follows. In Section 2 we formulate our main result. In Sections 3 we prove the finite speed propagations property to some time which depends on the parameters of the problem and the initial electromagnetic field. The method of proof is connected with nonhomogeneous variants of Stampacchia lemma, in fact, it is an adaptation of local energy or Saint–Venant principle like estimates method. Appendix A contains necessary interpolation inequalities and important properties of nonhomogeneous functional inequalities.

2. Main results

We introduce the following concept of generalized solution of the system (M):

Definition 2.1. Let $n > 1$, $p > 1$, $-p < m < p(n-1)$ and $w = E^2 + H^2$. A pair $(\mathbf{E}(x,t), \mathbf{H}(x,t))$ such that

$$w \in C(0,T; L^1(R^N)), \ w^{\frac{m+p}{p}} \in L^p(0,T; W^{1,p}(R^N)), \ w_t \in L^1(Q_T)$$

is called a solution of problem (M) if for a.e. $t > 0$ the integral identities

$$\frac{1}{2} \int_{R^N} E^2(t,x)\eta(t,x)\mathrm{d}x - \frac{1}{2} \iint_{Q_T} E^2(t,x)\eta_t(t,x)\mathrm{d}x\mathrm{d}t + \iint_{Q_T} a_1 E^2(t,x)\eta(t,x)\mathrm{d}x\mathrm{d}t -$$

$$-\iint_{Q_T} b_1 \mathbf{E}\,\mathrm{curl}\,\mathbf{H}\,\mathrm{d}x\mathrm{d}t = \frac{1}{2} \int_{R^N} E^2(0,x)\eta(0,x)\mathrm{d}x, \qquad (2.1)$$

$$\frac{1}{2} \int_{R^N} H^2(t,x)\eta(t,x)\mathrm{d}x - \frac{1}{2} \iint_{Q_T} H^2(t,x)\eta_t(t,x)\mathrm{d}x\mathrm{d}t + \iint_{Q_T} a_2 H^2(t,x)\eta(t,x)\mathrm{d}x\mathrm{d}t +$$

$$+\iint_{Q_T} b_2 \mathbf{H}\,\mathrm{curl}\,\mathbf{E}\,\mathrm{d}x\mathrm{d}t = \frac{1}{2} \int_{R^N} H^2(0,x)\eta(0,x)\mathrm{d}x \qquad (2.2)$$

are satisfied for every $\eta \in C^1(Q_T)$.

The main result is the following.

Theorem 1. Let the pair $(\mathbf{E}(x,t), \mathbf{H}(x,t))$ be a solution of the problem (M), in the sense of Definition 2.1. Let $p > 1, n > 1$ (and $n < 1 + \dfrac{(p-1)(p+N)}{pN(2-p)}$ if $p < 2$), and

$$\max\left\{-p, \ -p\left(1+\frac{1}{N}-\frac{n}{p}\right), \ -p\left(1+\frac{1}{N}-\frac{n-1}{p-1}\right)\right\} < m < p(n-2)+1.$$

Then there exists a time $T^* > 0$, depending on the known parameters only (in particular, $\square w(x,0)\square_{L_1(R^N)}$), and a function $\Gamma(t) \in C[0,T]$, $\Gamma(0) = 0$ such that

$$\Gamma(t) = K \max\left\{t^{\frac{p+N(m+p-n)}{p+N(m+p-1)}}, t^{\kappa}\right\} = K \begin{cases} t^{\kappa} \text{ for } t \leq 1, \\ t^{\frac{p+N(m+p-n)}{p+N(m+p-1)}} \text{ for } t > 1, \end{cases} \qquad (2.3)$$

where

$$\kappa = \frac{p\big(p-1+N(m+p-n)\big)\big[np+N(m+p-1)\big]}{\big(p+N(m+p-1)\big)\big[p(p(n-1)-m)+N(p-1)(m+p-1)\big]},$$

and

$$\mathrm{supp}\,w(t,.) \subset \left\{x = (x', x_N) \in R^N : x_N < \Gamma(t)\right\} \quad \forall\, 0 < t < T^*, \qquad (2.4)$$

i.e. $\mathbf{E}(x,t) = \mathbf{H}(x,t) = 0$ for all $x \in \left\{ x = (x', x_N) \in R^N : x_N < \Gamma(t) \right\}$. Here $K = K(n, m, p, N, \| w(0,x) \|_{L_1(R^N)})$ is a positive constant.

Remark 2.1. The statement of Theorem 1 stays true if we consider the problem for system (M) in some bounded domain. Then, instead of (1.25), we suppose that a support of initial energy of electromagnetic field is contained in some ball into the domain.

3. Proof of finite speed of propagations

Summing (2.1) and (2.2), in view of conditions (1.17) and (1.18), we find that

$$\frac{1}{2} \int_{R^N} w(t,x)\eta(t,x)\mathrm{d}x - \frac{1}{2} \iint_{Q_T} w(t,x)\eta_t(t,x)\mathrm{d}x\mathrm{d}t + d_1 \iint_{Q_T} w^m |\nabla w|^p \, \eta(t,x)\mathrm{d}x\mathrm{d}t +$$

$$+ \iint_{Q_T} b_1 \mathrm{div}(\mathbf{E} \times \mathbf{H})\eta(x,t)\mathrm{d}x\mathrm{d}t \leq \frac{1}{2} \int_{R^N} w(0,x)\eta(0,x)\mathrm{d}x . \tag{3.1}$$

Above we used the following relation:

$$\mathrm{div}(\mathbf{E} \times \mathbf{H}) = \mathbf{H} \, \mathrm{curl}\,\mathbf{E} - \mathbf{E} \, \mathrm{curl}\,\mathbf{H} . \tag{3.2}$$

From (3.1), (1.19) and (1.20) we get

$$\int_{R^N} w(x,T)\eta(x,T)\mathrm{d}x - \iint_{Q_T} w(x,t)\eta_t(x,t)\mathrm{d}x\mathrm{d}t + c\iint_{Q_T} |\nabla w^{\frac{m+p}{p}}|^p \, \eta(x,t)\mathrm{d}x\,\mathrm{d}t \leq$$

$$\leq \int_{R^N} w(x,0)\eta(x,0)\mathrm{d}x + 2d_2 \iint_{Q_T} w^{n-1} |\mathbf{E} \times \mathbf{H}| |\nabla \eta(x,t)| \, \mathrm{d}x\mathrm{d}t +$$

$$+ 2d_3 \iint_{Q_T} w^{n-2} |\nabla w| |\mathbf{E} \times \mathbf{H}| \eta(x,t)\mathrm{d}x\mathrm{d}t \leq \int_{R^N} w(x,0)\eta(x,0)\mathrm{d}x +$$

$$+ \varepsilon \iint_{Q_T} |\nabla w^{\frac{m+p}{p}}|^p \, \eta(x,t)\mathrm{d}x\mathrm{d}t + c(\varepsilon) \iint_{Q_T} w^{\frac{p(n-1)-m}{p-1}} \eta(x,t)\mathrm{d}x\mathrm{d}t +$$

$$+ c\iint_{Q_T} w^n |\nabla \eta(x,t)| \, \mathrm{d}x\mathrm{d}t \tag{3.3}$$

for every nonnegative function $\eta(x,t) \in C^1(Q_T)$, where $\varepsilon > 0, p > 1, n > 1$, $-p < m < p(n-2)+1$ (i.e. $\frac{p(n-1)-m}{p-1} > 1$).

For an arbitrary $s \in R^1$ and $\delta > 0$ we consider the families of sets

$$\Omega(s) = \left\{ x = (x', x_N) \in R^N : x_N \geq s \right\}, \quad Q_T(s) = (0,T) \times \Omega(s),$$

$$K(s,\delta) = \Omega(s) \setminus \Omega(s+\delta), \quad K_T(s,\delta) = (0,T) \times K(s,\delta).$$

Next we introduce our main cut-off functions $\eta_{s,\delta}(x) \in C^1(R^N)$ such that $0 \le \eta_{s,\delta}(x) \le 1 \ \forall x \in R^N$ and possess the following properties:

$$\eta_{s,\delta}(x) = \begin{cases} 0, x \in R^N \setminus \Omega(s), \\ 1, x \in \Omega(s+\delta), \end{cases} \qquad |\nabla \eta_{s,\delta}| \le \frac{c}{\delta} \quad \forall x \in K(s,\delta).$$

Choosing $\varepsilon > 0$ sufficiently small and

$$\eta(x,t) = \eta_{s,\delta}(x) \exp\left(-tT^{-1}\right) \quad \forall T > 0 \tag{3.4}$$

in integral inequality (3.3), we find

$$\sup_{t \in (0,T)} \int_{\Omega(s+\delta)} w(x,t)\mathrm{d}x + \frac{1}{T} \iint_{Q_T(s+\delta)} w(x,t)\mathrm{d}x\mathrm{d}t + c \iint_{Q_T(s+\delta)} |\nabla w^{\frac{m+p}{p}}|^p \, \mathrm{d}x\mathrm{d}t \le$$

$$\le \int_{\Omega(s)} w(x,0)\mathrm{d}x + \frac{c}{\delta} \iint_{K_T(s,\delta)} w^n \mathrm{d}x\mathrm{d}t + c \iint_{Q_T(s)} w^{\frac{p(n-1)-m}{p-1}} \mathrm{d}x\mathrm{d}t =: R_T(s,\delta), \tag{3.5}$$

where $s \in R^1, \delta > 0, T > 0$. Owing to (1.25), we have

$$\int_{\Omega(s)} w(x,0)\mathrm{d}x \equiv 0 \quad \forall s \ge 0. \tag{3.6}$$

We introduce the functions related to $w(x,t)$:

$$A_T(s) := \iint_{Q_T(s)} w^n \mathrm{d}x\mathrm{d}t, \quad B_T(s) := \iint_{Q_T(s)} w^{\frac{p(n-1)-m}{p-1}} \mathrm{d}x\mathrm{d}t.$$

Applying the interpolation inequality of Lemma A.2 in the domain $\Omega(s+\delta)$ to the function $v = w^{\frac{m+p}{p}}$ for $a = \frac{np}{m+p}$, $d = p$, $b = \frac{p}{m+p}$, $i = 0$, $j = 1$, and integrating the result with respect to time from 0 to T, we obtain

$$A_T(s+\delta) \le c\, T^{1-k_1} R_T^{1+\beta_1}(s,\delta), \tag{3.7}$$

where $k_1 = \dfrac{N(n-1)}{p+N(m+p-1)} < 1$, $\beta_1 = \dfrac{p(n-1)}{p+N(m+p-1)}$, $m > n - p\left(1+\dfrac{1}{N}\right)$. Similarly, applying the interpolation inequality of Lemma A.2 in the domain $\Omega(s+\delta)$ to the function $v = w^{\frac{m+p}{p}}$ for $a = \dfrac{p(p(n-1)-m)}{(p-1)(m+p)}$, $d = p$, $b = \dfrac{p}{m+p}$, $i = 0$, $j = 1$, and integrating the result with respect to time, we find that

$$B_T(s+\delta) \le c\, T^{1-k_2} R_T^{1+\beta_2}(s,\delta), \tag{3.8}$$

where $k_2 = \dfrac{N\big(p(n-2)-m+1\big)}{(p-1)\big(p+N(m+p-1)\big)} < 1$, $\beta_2 = \dfrac{p\big(p(n-2)-m+1\big)}{(p-1)\big(p+N(m+p-1)\big)}$, $m >$

$> \dfrac{p(n-1)}{p-1} - p\left(1+\dfrac{1}{N}\right)$. Next we define the function

$$C_T(s) := (A_T(s))^{1+\beta_2} + (B_T(s))^{1+\beta_1}.$$

Then

$$C_T(s+\delta) \le c\, F(T)[\delta^{-\beta} C_T^{1+\beta_1}(s) + C_T^{1+\beta_2}(s)], \tag{3.9}$$

where

$$\beta = (1+\beta_1)(1+\beta_2), \quad F(T) = \max\left\{ T^{(1-k_1)(1+\beta_2)}, T^{(1-k_2)(1+\beta_1)} \right\}.$$

Below, we find some estimate L^1-norm of $w(x,t)$ by L^1-norm of $w(x,0)$, which will be used in the next consideration.

Lemma 3.1. There exists some constant $c > 0$, depending on known parameters of the problem, such that the following estimate

$$\int_{R^N} w(x,t)\mathrm{d}x \le c \int_{R^N} w(x,0)\mathrm{d}x \quad \forall\, t \le T_1, \tag{3.10}$$

is valid. Here T_1 depends on m, p, n, N and $\|w(x,0)\|_{L^1(R^N)}$.

Proof. We set $s = -2\delta$, $\delta = s' > 0$ in (3.5) and pass to the limit as $s' \to \infty$

$$\sup_{t\in(0,T)} \int_{R^N} w(x,t)\mathrm{d}x + \frac{1}{T}\iint_{Q_T} w(x,t)\mathrm{d}x\mathrm{d}t + c\iint_{Q_T} |\nabla w^{\frac{m+p}{p}}|^p\,\mathrm{d}x\mathrm{d}t \le$$

$$\le \int_{R^N} w(x,0)\mathrm{d}x + \iint_{Q_T} w^{\frac{p(n-1)-m}{p-1}}\mathrm{d}x\mathrm{d}t. \tag{3.11}$$

Applying the interpolation inequality of Lemma A.2 in R^N to the function $v = w^{\frac{m+p}{p}}$ for $a = \dfrac{p\big(p(n-1)-m\big)}{(m+p)(p-1)}, d = p, b = \dfrac{p}{m+p}$, $i = 0, j = 1$, and Young's inequality, we find that

$$\int_{R^N} w^{\frac{p(n-1)-m}{p-1}}\mathrm{d}x \le c\left(\int_{R^N} |\nabla w^{\frac{m+p}{p}}|^p\,\mathrm{d}x\right)^{\frac{a\theta}{p}}\left(\int_{R^N} w\mathrm{d}x\right)^{\frac{a(1-\theta)}{b}} \le$$

$$\le \varepsilon \int_{R^N} |\nabla w^{\frac{m+p}{p}}|^p\,\mathrm{d}x + c(\varepsilon)\left(\int_{R^N} w\mathrm{d}x\right)^{\frac{ap(1-\theta)}{b(p-a\theta)}} \quad \forall\, \varepsilon > 0,$$

where $\theta = \dfrac{N(m+n)\big(p(n-2)-m+1\big)}{\big(N(m+p-1)+p\big)\big(p(n-1)-m\big)}$. Integrating this inequality with respect to time from 0 to T, we obtain

$$\iint\limits_{Q_T} w^{\frac{p(n-1)-m}{p-1}} \, dxdt \le \varepsilon \iint\limits_{Q_T} |\nabla w^{\frac{m+p}{p}}|^p \, dx + c(\varepsilon)\int\limits_0^T \left(\int\limits_{R^N} w dx\right)^{\frac{ap(1-\theta)}{b(p-a\theta)}} dt . \qquad (3.12)$$

Choosing $\varepsilon > 0$ sufficiently small, from (3.11), (3.12) we have

$$\sup\limits_{t\in(0,T)} \int\limits_{R^N} w(x,t)dx + \frac{1}{T}\iint\limits_{Q_T} w(x,t)dxdt + c\iint\limits_{Q_T} |\nabla w^{\frac{m+p}{p}}|^p \, dxdt \le$$

$$\le \int\limits_{R^N} w(x,0)dx + c\int\limits_0^T \left(\int\limits_{R^N} w dx\right)^{\frac{ap(1-\theta)}{b(p-a\theta)}} dt . \qquad (3.13)$$

From the last inequality we deduce that for every $t : 0 < t < T$ the following inequality is valid

$$\int\limits_{R^N} w(x,t)dx \le \int\limits_{R^N} w(x,0)dx + c\int\limits_0^t \left(\int\limits_{R^N} w(x,\tau)dx\right)^{\gamma} d\tau ,$$

where $\gamma = \dfrac{(N-1)\big(p(n-1)-m\big)+N(p-1)(m+p)}{p\big(p-1+N(m+p-n)\big)}$. Applying Lemma A.3 from Appendix A we obtain (3.10) with

$$T_1 := \begin{cases} \dfrac{2}{1-\gamma}\left(\displaystyle\int\limits_{R^N} w(x,0)dx\right)^{1-\gamma} & \text{if } \gamma < 1, \\[4mm] \dfrac{1}{2(\gamma-1)}\left(\displaystyle\int\limits_{R^N} w(x,0)dx\right)^{\gamma-1} & \text{if } \gamma > 1, \end{cases} \qquad (3.14)$$

and $T_1 \to 0$ as $\| w(x,0)\|_{L_1(R^N)} \to 0$.

Further, using the definition of the functions $C_T(s)$ and (3.10), we get

$$C_T(s_0) \le K_0 F(T) \qquad \forall T \le T_1, \qquad (3.15)$$

where the positive constant K_0 depends on n, m, p, N and $\| w(x,0)\|_{L^1(R^N)}$.

Now we choose the parameter $\delta > 0$ which was arbitrary up to now:

$$\delta_T(s) := \left[\frac{2c}{1-H_T(s_0)} F(T) C_T^{\beta_1}(s)\right]^{\frac{1}{\beta}} ,$$

where the function $H_T(s) = cF(T)C_T^{\beta_2}(s)$ is such that $H_T(s_0) < 1$ at some point $s_0 \geq 0$, whence we obtain

$$T \leq T_2 = c \min\left\{ K_0^{-\frac{\beta_2}{(1-k_1)(1+\beta_2)^2}}, K_0^{-\frac{\beta_2}{(1-k_2)(1+\beta_1)(1+\beta_2)}} \right\}, \qquad (3.16)$$

and $T_2 \to \infty$ as $\Vert w(x,0) \Vert_{L_1(R^N)} \to 0$.

We obtain the following main functional relation for the function $\delta_T(s)$:

$$\delta_T(s + \delta_T(s)) \leq \varepsilon \delta_T(s) \quad \forall s \geq s_0 \geq 0, \; 0 < \varepsilon = \left(\frac{1 + H_T(s_0)}{2} \right)^{\frac{\beta_1}{\beta}} < 1 \quad (3.17)$$

$\forall \; 0 < T < T^* := \min\{T_1, T_2\}$, with T_1 of (3.14) and T_2 of (3.16). Now we apply Lemma A.1 to the function $\delta_T(s)$ of (3.17). As a result, we obtain

$$\delta_T(s) \equiv 0 \quad \forall s \geq s_0 + \frac{1}{1-\varepsilon} \delta_T(s_0). \qquad (3.18)$$

Then, in view of (3.15), we find

$$\delta_T(s_0) \leq c \left[C_T^{\beta_1}(s_0) F(T) \right]^{\frac{1}{\beta}} \leq c [F^{1+\beta_1}(T)]^{\frac{1}{\beta}} \leq c \left(F(T) \right)^{\frac{1}{1+\beta_2}} = c \max\left\{ T^{1-k_1}, T^{\frac{(1-k_2)(1+\beta_1)}{1+\beta_2}} \right\}$$

$\forall 0 < T < T^*$. Choosing in (3.18) $s_0 = 0$ and

$$s = \Gamma(T) = c \max\left\{ T^{\frac{p+N(m+p-n)}{p+N(m+p-1)}}, T^\kappa \right\} = c \begin{cases} T^\kappa & \text{for } T < 1, \\ T^{\frac{p+N(m+p-n)}{p+N(m+p-1)}} & \text{for } T > 1 \end{cases}$$

$\forall 0 < T < T^*$, $\kappa = \dfrac{p\big(p-1+N(m+p-n)\big)\big[np+N(m+p-1)\big]}{\big(p+N(m+p-1)\big)\big[p(p(n-1)-m)+N(p-1)(m+p-1)\big]}$. Thus

$w(T,x) \equiv 0$ for all $x \in \{x = (x', x_N) \in R^N : x_N \geq \Gamma(t)\}$. And Theorem 1 is proved completely.

Appendix A

Lemma A.1. [22] Let the nonnegative continuous nonincreasing function $f(s)$: $[s_0, \infty) \to R^1$ satisfy the following functional relation:

$$f(s + f(s)) \leq \varepsilon f(s) \quad \forall s \geq s_0, \; 0 < \varepsilon < 1.$$

Then $f(s) \equiv 0 \ \forall \ s \geq s_0 + (1-\varepsilon)^{-1} f(s_0)$.

Lemma A.2. [18] If $\Omega \subset R^N$ is a bounded domain with piecewise-smooth boundary, $a > 1$, $b \in (0,a), d > 1$, and $0 \leq i < j$, $i, j \in N$, then there exist positive constants d_1 and d_2 ($d_2 = 0$ if the domain Ω is unbounded) that depend only on $\Omega, d, j, b,$ and N and are such that, for any function $v(x) \in W_d^j(\Omega) \cap L^b(\Omega)$, the following inequality is true:

$$\left\| D^i v \right\|_{L^a(\Omega)} \leq d_1 \left\| D^j v \right\|_{L^d(\Omega)}^{\theta} \|v\|_{L^b(\Omega)}^{1-\theta} + d_2 \|v\|_{L^b(\Omega)}$$

where $\theta = \dfrac{\dfrac{1}{b} + \dfrac{i}{N} - \dfrac{1}{a}}{\dfrac{1}{b} + \dfrac{j}{N} - \dfrac{1}{d}} \in \left[\dfrac{i}{j}, 1 \right)$.

Lemma A.3. [1] Suppose that v(t) is a nonnegative summable function on [0,T] that, for almost all $t \in [0,T]$, satisfies the integral inequality

$$v(t) \leq k + m \int_0^t h(\tau) g\big(v(\tau)\big) \mathrm{d}\tau,$$

where $k \geq 0$, $m \geq 0$, $h(\tau)$ is summable on $[0,T]$, and $g(\tau)$ is a positive function for $\tau > 0$. Then

$$v(t) \leq G^{-1}\left(G(k) + m \int_0^t h(\tau) \mathrm{d}\tau \right)$$

for almost all $t \in [0,T]$. Here $G(v) = \int_{v_0}^v \dfrac{\mathrm{d}\tau}{g(\tau)}, v > v_0 > 0$.

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СКІНЧЕНА ШВИДКІСТЬ ПОШИРЕННЯ ЗБУРЕНЬ ЕЛЕКТРОМАГНІТНОГО ПОЛЯ У НЕЛІНІЙНИХ ІЗОТРОПНИХ ДИСПЕРСНИХ СЕРЕДОВИЩАХ

Запропоновано модифікацію рівнянь Максвела, що описує середовища, в яких електричні та магнітні властивості суттєво змінюються під впливом зовнішнього електромагнітного поля. Для таких середовищ встановлено, що електромагнітні хвилі розповсюджуються зі скінченою швидкістю на протязі часу, який, в свою чергу, залежить від початкової енергії електромагнітного поля та нелінійних параметрів середовища.

Ключові слова: рівняння Максвела, нелінійне дисперсне середовище, скінчена швидкість поширення збурень, асимптотична поведінка

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КОНЕЧНАЯ СКОРОСТЬ РАСПРОСТРАНЕНИЯ ВОЗМУЩЕНИЙ ЭЛЕКТРОМАГНИТНОГО ПОЛЯ В НЕЛИНЕЙНЫХ ИЗОТРОПНЫХ ДИСПЕРСИОННЫХ СРЕДАХ

Предложена модификация уравнений Максвелла, описывающая среды, в которых электрические и магнитные свойства существенно изменяются под воздействием внешнего электромагнитного поля. Для таких сред установлено, что электромагнитные волны распространяются с конечной скоростью в течение времени, зависящего от начальной энергии электромагнитного поля и нелинейных параметров среды.

Ключевые слова: уравнения Максвелла, нелинейная дисперсная среда, конечная скорость распространения возмущений, асимптотическое поведение